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# The generalized modified shift-splitting preconditioners for nonsymmetric saddle point problems☆

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ABSTRACT

For a nonsymmetric saddle point problem, the modified shift-splitting (MSS) preconditioner has been proposed by Zhou et al. By replacing the parameter $\alpha$ in (2,2)-block in the MSS preconditioner by another parameter $\beta$, a generalized MSS (GMSS) preconditioner is established in this paper, which results in a fixed point iteration called the GMSS iteration method. We provide the convergent and semi-convergent analysis of the GMSS iteration method, which show that this method is convergence and semi-convergence if the related parameters satisfy suitable restrictions. Meanwhile, the distribution of eigenvalues and the forms of the eigenvectors of the preconditioned matrix are analyzed in detail. Finally, numerical examples show that the GMSS method is more feasibility and robustness than the MSS, Uzawa-HSS and PU-STS methods as a solver, and the GMSS preconditioner outperforms the GSOR, Uzawa-HSS, MSS and LMSS preconditioners for the GMRES method for solving both the nonsingular and the singular saddle point problems with nonsymmetric positive definite and symmetric dominant (1,1) parts.



## 1. Introduction

Consider the following large and sparse saddle point problem

$$\mathcal{A}u = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ is nonsymmetric positive definite, $B \in \mathbb{R}^{m\times n}$, $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$ with $n \le m$. It follows that the saddle point problem (1) is nonsingular when $B$ is of full column rank and singular when $B$ is rank deficient [16].

The saddle point problem (1) is important and arises in a variety of scientific and engineering applications, such as mixed finite element approximation of elliptic partial differential equations, optimal control, computational fluid dynamics, weighted least-squares problems, electronic networks, computer graphics etc; see [2,16,20] and references therein.

When $B$ in (1) is of full rank, i.e., the saddle point problem (1) is nonsingular, a number of iteration methods and their numerical properties have been discussed to solve the saddle point problem (1) in the literature, such as SOR-like methods [12,14,28,29], Uzawa-type methods [12,14,19,27,37,46], Hermitian and skew-Hermitian splitting (HSS) methods [8] and

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its variants [3,5,7,9,10], RPCG iteration methods [11,13,44] and Krylov subspace iteration methods [39] with high-quality preconditioners [2,16].

Though most often the matrix $B$ occurs in the form of full column rank, but not always in practice. For example, in the finite difference discretization of the Navier–Stokes equation with periodic boundary conditions, $B$ in (1) becomes singular. When $B$ in (1) is rank deficient, the saddle point problem (1) is singular. In recent years, there has been a surge of interest in solving singular saddle point problem (1). In [33,45,47,48], the authors applied the Uzawa-type methods to solve singular saddle point problems. Yang et al. [42] discussed the Uzawa-HSS method for singular saddle point problems. Wang and Zhang [41] presented the preconditioned AHSS iteration method for singular saddle point problems. Chen and Ma [25] and Cao and Miao [23] investigated the generalized shift-splitting iteration method for singular saddle point problems. Very recently, Liang and Zhang [36] developed the convergence behavior of generalized parameterized Uzawa method for singular saddle point problems.

Recently, based on the well-known Hermitian and skew-Hermitian splitting (HSS) of the matrix $A$ as follows

$$A = H + S,$$

where $H = \frac{1}{2}(A + A^T)$, $S = \frac{1}{2}(A - A^T)$, and similar to the shift-splitting [15,21], the modified shift-splitting (MSS) preconditioner [49] was proposed for nonsymmetric saddle point problem (1), i.e., $\mathcal{A} = \mathcal{P}_{MSS} - \mathcal{Q}_{MSS}$, where

$$\mathcal{A} = \mathcal{P}_{MSS} - \mathcal{Q}_{MSS} = \frac{1}{2}\begin{pmatrix} \alpha I + 2H & B \\ -B^T & \alpha I \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I - 2S & -B \\ B^T & \alpha I \end{pmatrix} \tag{2}$$

with $\alpha > 0$ being a constant and $I$ being the identity matrix with appropriate dimension. Based on the splitting (2), the MSS iteration method is constructed for saddle point problems as follows:

*The MSS iteration method:* given initial guess $x^{(0)} \in \mathbb{R}^m$ and $y^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots$, until the iteration sequence $\{(x^{(k)^T}), (y^{(k)^T})\}$ is convergent, the matrix form of the MSS iteration algorithm is:

$$\frac{1}{2}\begin{pmatrix} \alpha I + 2H & B \\ -B^T & \alpha I \end{pmatrix}\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha I - 2S & -B \\ B^T & \alpha I \end{pmatrix}\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{3}$$

where $\alpha$ is a given positive constant.

In this paper, the idea of the MSS preconditioner is generalized and a generalized MSS (GMSS) preconditioner for the saddle point problem (1) is established. In the meanwhile, the convergence and semi-convergence of the GMSS iteration method are studied. Besides, the spectral properties of the GMSS preconditioned matrix are investigated. Numerical experiments are presented to show the effectiveness of the GMSS iteration method and the GMRES method with the GMSS preconditioner for solving the saddle point problems.

The remainder of this paper is organized as follows. In Section 2, we propose the generalized shift-splitting (GMSS) preconditioner and derive the implementation of this new preconditioner. The convergence properties of the GMSS iteration method for solving nonsingular saddle point problems and the choice of the iteration parameters are discussed in Section 3. The semi-convergence conditions of the GMSS iteration method for solving singular saddle point problems will be given in Section 4. The spectral properties of the GMSS preconditioned matrix are described in Section 5. Section 6 is devoted to performing numerical examples to examine the feasibility and effectiveness of the GMSS iteration method and the GMSS preconditioned GMRES method for solving the saddle point problems. Finally, the paper is ended with some conclusions in Section 7.

## 2. The generalized modified shift-splitting preconditioner and its implementation

To obtain the new iteration methods for saddle point problems, some authors introduced new parameters in the existing methods and constructed the better methods [5,30,32,34,40]. Based on the preconditioned HSS (PHSS) method derived by Bai et al. [10], Bai and Golub [5] and Li et al. [34] obtained the AHSS and parameterized preconditioned HSS (PPHSS) methods, respectively. Recently, on the basis of the shift-splitting (SS) preconditioner [21], Chen and Ma [24] and Cao et al. [22] replaced the parameter $\alpha$ in (2,2)-block of the SS preconditioner by another parameter $\beta$, and employed the generalized SS (GSS) preconditioner. Numerical experiments were provided to demonstrate that the GSS preconditioner outperforms the SS preconditioner. This idea motivates us to develop the generalized modified shift-splitting (GMSS) preconditioner for saddle point problem (1) by replacing the parameter $\alpha$ in (2,2)-block in the MSS preconditioner by another parameter $\beta$. Since the parameters $\alpha$ and $\beta$ in (1,1) and (2,2)-blocks of the GMSS preconditioner are not associated, it is anticipated that the GMSS iteration method and the GMSS preconditioner with proper chosen parameters would be much better than the MSS iteration method given by (3) and the MSS preconditioner, respectively.

Let $A = H + S$ be the symmetric and skew-symmetric splitting of the (1,1)-block of the matrix $\mathcal{A}$ defined as in (1), where $H = (A + A^T)/2$ and $S = (A - A^T)/2$. Based on the MSS iteration method, we replace the parameter $\alpha$ in (2,2)-block of the MSS preconditioner by another parameter $\beta$ with $\beta > 0$, and a new splitting which is referred to as the generalized modified shift-splitting (GMSS) for the nonsymmetric matrix $\mathcal{A}$ is derived as follows

$$\mathcal{A} = \mathcal{P}_{GMSS} - \mathcal{Q}_{GMSS} = \frac{1}{2}\begin{pmatrix} \alpha I + 2H & B \\ -B^T & \beta I \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I - 2S & -B \\ B^T & \beta I \end{pmatrix}, \tag{4}$$