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Convergence analysis of the Jacobi-collocation method for nonlinear weakly singular Volterra integral equations

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ABSTRACT

In this work, we present an efficient spectral-collocation method for numerical solution of a class of nonlinear weakly singular Volterra integral equations. This type of equations typically has a singular behavior at the left endpoint of the interval of integration. For overcoming this non-smooth behavior, we apply the Jacobi-collocation method. The convergence analysis of the proposed method is investigated in the L^∞ and the weighted L^2 norms and the results of several numerical experiments are presented which support the theoretical results. The computed results are compared wherever possible with those already available in the literature.

1. Introduction

The singular integral equations are often encountered in many problems of mechanics, engineering [1,2], physics and chemical reactions, such as heat conduction, crystal growth and superfluidity [3]. In this paper, we consider the nonlinear weakly singular Volterra integral equation of the second kind of the form

$$y(t) = g(t) + \int_0^t \frac{K(t,s,y(s))}{(t-s)^\mu} ds, \quad 0 \le t \le T, \tag{1.1}$$

where $0 < \mu < 1$, K and g are given functions, and y is an unknown function to be determined. We assume that $K(t, s, y(s))$ satisfies the Lipschitz condition, i.e., for fixed t, s with $0 \le s \le t \le T$, there exists a positive constant L, independent of t and s, such that

$$|K(t,s,u) - K(t,s,v)| \le L|u - v|, \quad \forall u, v \in (-\infty, \infty). \tag{1.2}$$

The existence of solutions and the regularity results of some singular integral equations have been considered in some references such as [4–6].

Several authors have written a number of papers which establish numerical techniques for finding an approximation of the linear and nonlinear weakly singular Volterra integral equations [7–19]. We will briefly review some of these techniques. In [12], Brunner et al. analyzed piecewise polynomial collocation method for nonlinear weakly singular Volterra equations with logarithmic and algebraic kernels. Baratella and Orsi [13] studied standard product integration methods to solve a nonlinear weakly singular Volterra integral equation. Moreover, Tao and Yong [14] used extrapolation method to solve weakly

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singular nonlinear Volterra integral equations of the second kind. For more research works on nonlinear weakly singular Volterra integral equations, see [15–17].

The main goal of this paper is to present a high order numerical method for solving Eq. (1.1) by using Jacobi-collocation method. We use some function and variable transformations to change the equation into a new Volterra integral equation which possesses better regularity. When Eq. (1.1) is linear, by using a variable transformation, it changes into an equation which is still weakly singular, but whose solution is as smooth as we like [13]. In the nonlinear case, we apply the following transformation [13,14]

$$\gamma(s) = s^q, \tag{1.3}$$

to change (1.1) into the following equation

$$\begin{aligned} &y(\gamma(t)) = g(\gamma(t)) + \int_0^t (\gamma(t)-\gamma(s))^{-\mu} K(\gamma(t),\gamma(s),y(\gamma(s)))\gamma'(s)ds,\\ &\gamma^{-1}(0) = 0 \le s \le t \le \sqrt[q]{T} = \gamma^{-1}(T), \end{aligned} \tag{1.4}$$

where q is a positive integer. Let

$$\hat{y}(t) = y(\gamma(t)), \quad \hat{g}(t) = g(\gamma(t)), \tag{1.5}$$

then, from Eq. (1.4) we have

$$\hat{y}(t) = \hat{g}(t) + \int_0^t (\gamma(t)-\gamma(s))^{-\mu} K(\gamma(t),\gamma(s),\hat{y}(s))\gamma'(s)ds. \tag{1.6}$$

Now, we rewrite Eq. (1.6) as

$$\hat{y}(t) = \hat{g}(t) + \int_0^t \hat{K}(t,s,\hat{y}(s))ds, \tag{1.7}$$

where

$$\hat{K}(t,s,\hat{y}(s)) = (t-s)^{-\mu}\bar{K}(t,s,\hat{y}(s)), \tag{1.8}$$

$$\bar{K}(t,s,\hat{y}(s)) = \begin{cases} \left(\dfrac{\gamma(t)-\gamma(s)}{t-s}\right)^{-\mu} K(\gamma(t),\gamma(s),\hat{y}(s))\gamma'(s), & t \ne s,\\ (\gamma'(t))^{-\mu} K(\gamma(t),\gamma(s),\hat{y}(s))\gamma'(s), & t = s, \end{cases} \tag{1.9}$$

and $\bar{K}(t,s,\hat{y}(s))$ is a smooth function. Now by choosing a suitable q, we can see that the solution of Eq. (1.7) is sufficiently smooth (see [14] for further details) and the Jacobi spectral method can be applied conveniently.

The layout of the paper is as follows. In Section 2, the Jacobi-collocation method is used to approximate the solution of Eq. (1.1). As a result, a set of algebraic equations is achieved and the solution of the considered problem is introduced. In Section 3, we introduce some useful lemmas to establish the convergence results. We will illustrate the convergence analysis of the presented method in Section 4. In the last section, numerical examples are given to show accuracy, validity and applicability of the numerical technique.

2. Description of the method

Let $w^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, for $\alpha,\beta > -1$ denotes a weight function in the usual sense and $P_N(\Lambda)$ be the space of all polynomials with degree not exceeding N on Λ, where $\Lambda = [-1,1]$. It is well known, the set of Jacobi polynomials $\{J_N^{\alpha,\beta}\}_{N=0}^{\infty}$ forms a complete $L^2_{w^{\alpha,\beta}}(\Lambda)$ orthogonal system, where

$$L^2_{w^{\alpha,\beta}}(\Lambda) = \{f \,|\, f \text{ is measurable and } \|f\|_{L^2_{w^{\alpha,\beta}}} < \infty\}, \tag{2.1}$$

and

$$\|f\|^2_{L^2_{w^{\alpha,\beta}}} = (f,f)_{L^2_{w^{\alpha,\beta}}} = \int_{-1}^1 |f(x)|^2 w^{\alpha,\beta}(x)dx. \tag{2.2}$$

Now, let $H^m_{w^{\alpha,\beta}}$ denotes the Sobolev space of all functions $u(x)$ on Λ such that $u(x)$ and all its weak derivatives up to order m are in $L^2_{w^{\alpha,\beta}}(\Lambda)$, with the norm and the semi-norm as

$$\|u(x)\|^2_{H^m_{w^{\alpha,\beta}}(\Lambda)} = \sum_{k=0}^m \left\| \frac{\partial^k}{\partial x^k} u(x) \right\|^2_{L^2_{w^{\alpha,\beta}}(\Lambda)}, \tag{2.3}$$