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Nodal superconvergence of the local discontinuous Galerkin method for singularly perturbed problems

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ABSTRACT

In this paper, a superconvergence of order $(\ln N/N)^{2k+1}$ for the numerical traces of the LDG approximation to a one dimensional singularly perturbed convection–diffusion–reaction problem is proved. The LDG method is applied on a Shishkin mesh with $2N$ elements, and we use polynomials of degree at most k on each element. This result puts the numerical finding reported in Xie and Zhang (2007), Xie et al. (2009) on firm mathematical ground.

1. Introduction

We are interested in the singularly perturbed convection–diffusion–reaction problem

$$\begin{aligned} -\epsilon u'' + au' + bu &= f \qquad \text{in } \Omega = (0, 1), \\ u &= 0 \qquad \text{on } \partial\Omega = \{0, 1\}, \end{aligned} \tag{1.1}$$

where $0 < \epsilon \ll 1$ is the diffusion parameter, $a = a(x) \geq \alpha > 0$ accounts for the convection, and $b = b(x)$ accounts for the reaction term. The function $f = f(x)$ is a given source term. We assume that α is a constant; a, b, and f are sufficiently smooth on $\overline{\Omega}$ and satisfy

$$b - a'/2 \geq c_0 > 0, \quad or, \tag{1.2a}$$

$$b = 0, \quad a = constant \tag{1.2b}$$

for some constant c_0. The LDG method we consider will be based on the following system of first order differential equations that is equivalent to (1.1)

$$q = \epsilon u' \qquad \text{in } \ \Omega, \tag{1.3a}$$

$$-\, q' + au' + bu = f \qquad \text{in } \ \Omega, \tag{1.3b}$$

$$u = 0. \qquad \text{on } \partial\Omega. \tag{1.3c}$$

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It is well-known [1] that the exact solution u of (1.1) exhibits a boundary layer of width $O(\epsilon \ln \epsilon^{-1})$ at the outflow boundary $x = 1$. An efficient way of handling this problem from a numerical method perspective is employing a layer-adapted mesh, such as a Shishkin mesh. A considerable amount of literature has been devoted to theoretical analysis of such methods [1–7]. In recent years, various studies have been carried out for singularly perturbed convection–diffusion problems whose solutions exhibit boundary layers [8–14].

Among these methods the streamline diffusion finite element method (SDFEM) developed by Brooks and Hughes [15] exhibits a good performance in resolving boundary layers [2,16,17]. However, few studies have focused specifically on nodal superconvergence of numerical methods at meshes nodes. It is well known that Douglas and Dupont [18] proved that finite element approximation to Problem (1.3) using piecewise continuous polynomials of degree at most k converges with order $2k$ at the nodes of the mesh. This result was valid for arbitrary quasi-uniform meshes and the case $\epsilon = O(1)$. Indeed, numerical experiments showed that it is impossible to extend such results to the singularly-perturbed regime on arbitrary meshes. Recently, Celiker, Zhang, and Zhu in [8] proved that the numerical solution of SDFEM converges with order $(\ln N/N)^{2k}$ when Shishkin meshes are used, which was verified to be sharp by numerical experiments.

A nodal superconvergence of order $(\ln N/N)^{2k+1}$ for the LDG approximation on Shishkin meshes to solution u and its derivative for (1.1) was observed by Xie et al. and was reported in [10,11]. In this paper, we prove this nodal superconvergence result.

The outline of the rest of the paper is as follows. In Section 2, we recall the LDG method and state our main result. The proof of the main result is given in Section 3. We end in Section 4 with some concluding remarks.

2. The main result

In this section, we present the LDG discretization and state our main result. We begin with introducing the Shishkin mesh that will be used to approximate the solution of (1.1) using the LDG method. Let

$$\tau = \min\left\{\frac{1}{2}, \frac{\kappa}{\alpha}\epsilon \ \ln 2N\right\} \tag{2.1}$$

and set

$$H = \frac{1-\tau}{N} \quad \text{and} \quad h = \frac{\tau}{N}.$$

The number τ is called the *transition number* and κ is chosen as

$$\kappa = 2(k+1), \tag{2.2}$$

where $k \geq 1$ denotes degree of polynomials used on each element of the mesh. Furthermore, throughout the paper we always assume that N and ϵ are such that

$$\epsilon \leq CN^{-1} \tag{2.3}$$

for some constant C. It is reasonable to make this assumption, since, otherwise, the problem can be handled by uniform mesh.

The *nodes* of the mesh are defined recursively by setting $x_0 = 0$ and

$$x_j = \begin{cases} x_{j-1} + H, & \text{for } j = 1, \dots, N, \\ x_{j-1} + h, & \text{for } j = N+1, \dots, 2N. \end{cases}$$

The set of all nodes, $\{x_0, x_1, \dots, x_{2N}\}$, will be denoted by $\mathscr{E}_N$. Note that the node $x_N = 1 - \tau$ which is called the *transition point*. For $j = 1, 2, \dots, 2N$, we define $I_j = (x_{j-1}, x_j)$ as the jth element of the finite element mesh $\Omega_N = \bigcup_{j=1}^{2N} I_j$. Since the mesh Ω_N is piecewise uniform we define

$$\Omega_{\mathrm{R}} = \bigcup_{j=1}^{N} I_j \quad \text{and} \quad \Omega_{\mathrm{BL}} = \bigcup_{j=N+1}^{2N} I_j.$$

Clearly, Ω_{R} is a uniform discretization of the interval $(0, 1-\tau)$ of meshsize H, and Ω_{BL} is that of the interval $(1-\tau, 1)$ of meshsize h. The length of the jth element is defined as $h_j = x_j - x_{j-1}$.

The DG finite element space is defined as

$$V_N := \{v \in L^2(\Omega) : v|_I \in \mathcal{P}^k(I), \forall I \in \Omega_N\}$$

where $\mathcal{P}^k(I)$ is the space of polynomials of degree at most k on I. We will use the standard notation

$$(\varphi, \psi)_{\mathcal{D}} := \sum_{I \in \mathcal{D}} (\varphi, \psi)_I, \qquad \langle \varphi, \psi \rangle_{\partial \mathcal{D}} := \sum_{I \in \mathcal{D}} \langle \varphi, \psi \rangle_{\partial I}$$