Journal of Computational and Applied Mathematics 330 (2018) 117–127

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Geometric characteristics of planar quintic Pythagorean-hodograph curves

Lincong Fang [a,*], Guozhao Wang [b]

[a] *School of Information, Zhejiang University of Finance & Economics, 310018 Hangzhou, China*
[b] *School of Mathematical Sciences, Zhejiang University, 310027 Hangzhou, China*

ARTICLE INFO

Article history:
Received 14 September 2014
Received in revised form 20 August 2017

Keywords:
Geometric characteristic
Bézier curve
Pythagorean-hodograph
Control polygon

ABSTRACT

This study examines necessary and sufficient conditions for a planar quintic Bézier curve to be a Pythagorean-hodograph (PH) curve. Quintic PH curves can be categorized into two classes according to the representation of their derivatives. While the first class has been studied by Farouki (1994) to be a family of regular curves already, a more succinct proof by introducing auxiliary control points is provided in this paper. Geometric characteristics of the second class of quintic PH curves are also studied. The key technique to simplify the discussion is to represent a planar Bézier curve with a complex polynomial in Bernstein form. Benefiting from such complex expression, algebraic characteristics of quintic PH curves can be described by nonlinear complex systems with respect to control points. By treating these systems with geometric methods, conditions for a quintic planar curve to be a PH curve can be described in terms of geometric constraints on its control polygon. Furthermore, we provide methods for the construction of the second class of quintic PH curves. In particular, parameter values of cusps can be explicitly determined in advance for irregular curves.

1. Introduction

Pythagorean-hodograph (PH) curves introduced in [1] are an important class of polynomial curves that forms mathematical foundation of most current computer aided design (CAD) tools. By incorporating special algebraic structures in their tangent curves, PH curves possess a number of advanced properties over ordinary polynomial parametric curves. These properties include polynomial arc-length functions and rational offset curves. Hence, PH curves are considered as an elegant solution to a variety of difficult issues arising in applications (*e.g.*, tool paths) in the fields of computer numerical control machining and real-time motion control. For example, the arc-length of a PH curve can be computed without numerical integration, thus accelerating algorithms for numerically controlled (NC) machining [2]. The offsets of a PH curve can also be represented exactly rather than being approximated in CAD systems. Thus, analyzing and manipulating PH curves are of great practical value in CAD and other applications.

The concept of planar polynomial PH curve [1,3–5] is generalized to higher dimension spaces [6–9] and rational polynomial curves [10,11]. For more details about PH curves, we refer the readers to a comprehensive survey [12] and references therein.

PH curves can be represented as widely used Bézier curves in general, the most intuitive and efficient method for constructing PH curves is by adjusting the control points of Bézier curves under conditions guaranteeing PH properties.

* Corresponding author.
E-mail address: lincongfang@gmail.com (L. Fang).

http://dx.doi.org/10.1016/j.cam.2017.08.014
0377-0427/

Hence, intrinsic geometric characteristics of control polygons of PH curves should be investigated. For example, an intuitive geometric condition on the control polygon was proposed for a cubic Bézier curve to be a PH curve [1]. The planar PH cubic is considered to be unique if the freedoms of rigid motion, uniform scaling, and linear re-parametrization are not taken into account. To obtain more shape flexibility for practical application, PH curves of higher degrees are considered [3,13]. In particular, quintic PH curves have sufficient degree of freedoms for many applications. For example, quintic PH curves can satisfy general C^1 or G^2 geometric conditions [3,14], hence can be used to construct round corners between two given curves [14,15]. Some special curves (*e.g.*, spiral curves [16]) can also be designed by using PH quintics.

Although quintic PH curves exhibit greater shape flexibility, we can hardly derive a simple and all-encompassing characterization in terms of constraints on their control polygons. The constraints on the control polygons of a subset of quintic PH curves were discussed in [3,14]. In this paper, we go one step further to give a complete classification of quintic PH curves and derive geometric characteristics of each class in terms of constraints on their control polygons.

As will be seen in the following sections, quintic PH curves can be classified into two classes, referred as class I and class II curves, according to the factorizations of their tangent curves. The family of regular quintic PH curves studied in [3,17] is essentially the class I curves in the present paper. The authors in [3] provided a sufficient and necessary condition for a quintic Bézier curve to be a class I curve and we will express a proof more succinctly by introducing auxiliary control points. Besides, we discuss geometric characteristics of class II curves, which can be regular or irregular curves. To the best of our knowledge, this is the first work that provides geometric condition for class II curves. To achieve these results, we represent a planar Bézier curve by a complex polynomial in terms of Bernstein polynomials, which enables a clear and uniform analysis on geometric characteristics of both class I and class II curves.

Many recent works have used PH curves as specific tool for solving practical problems, such as speed re-parametrization, where Hermite interpolation problem is usually involved. Hence, there is a lot of research on the construction of a PH curve in Hermite interpolation problem with diverse boundary conditions [3,6–8,17–22]. As an application of our theoretical results, we show that a regular or irregular quintic PH curve interpolating given geometric data can be simply constructed by solving a nonlinear equation. It has been proved that there are four different class I regular quintic PH curves under a given C^1 Hermite interpolation condition [12,17,19]. We show that there are also four different class II quintic PH curves which satisfy a given C^1 Hermite interpolation condition. It has been known that an irregular quintic PH curve has two cusps. We show that an irregular quintic PH curve with cusps outside the interval of interest can also be constructed by solving complex equations.

The remainder of this paper is organized as follows: Section 2 introduces some preliminary definitions and notations of PH curves in Bernstein form. Section 3 categorizes quintic PH curves into two classes, which are discussed, respectively. Section 4 provides approaches for the construction of quintic PH curves under initial geometric conditions. Besides, some experiments are also conducted to validate the intuitiveness and effectiveness of our methods. Finally, Section 5 concludes the paper.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all real and complex numbers, respectively. Throughout this paper, we denote a complex number by a single bold character, *e.g.*, $\boldsymbol{z}$. For a complex number $\boldsymbol{z} \in \mathbb{C}$, we denote its conjugate complex number by $\bar{\boldsymbol{z}}$, and its complex norm on $\mathbb{C}$ by $\|\boldsymbol{z}\|$. Following [3,12], we use the complex representation of $\mathbb{R}^2$ to facilitate the derivation in the subsequent analysis of planar PH curves. Thus, a planar point is denoted by an order pair of real numbers (x, y) and a complex number $x + \mathbf{i}y$ interchangeably. Similarly, a planar parametric curve $\boldsymbol{P}(t) = (x(t), y(t))$, $t \in [0, 1]$, can be identified with a complex-valued function $\boldsymbol{P}(t) = x(t) + \mathbf{i}y(t)$, and vice versa.

Let $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$, $i = 0, \ldots, n$, be Bernstein polynomials, a quintic Bézier curve is defined by

$$\boldsymbol{P}(t) = \sum_{i=0}^{5} B_i^5(t)\boldsymbol{P}_i, \quad 0 \le t \le 1, \tag{1}$$

where $\boldsymbol{P}_i$, $i = 0, \ldots, 5$, are the control points. The polygon formed by consecutively connecting control points is called the *control polygon*. Let $\Delta\boldsymbol{P}_i$ denote the first forward-difference of the ith control point, *i.e.*, $\Delta\boldsymbol{P}_i = \boldsymbol{P}_{i+1} - \boldsymbol{P}_i$. Then, the derivative of the curve (1) can be represented in Bernstein form as

$$\boldsymbol{P}'(t) = 5\sum_{i=0}^{4} B_i^4(t)\Delta\boldsymbol{P}_i. \tag{2}$$

Let $L_i = \|\Delta\boldsymbol{P}_i\|$, then $\Delta\boldsymbol{P}_i$ can be expressed in polar coordinates as $\Delta\boldsymbol{P}_i = L_i e^{\mathbf{i}\phi_i}$ with certain $\phi_i \in [0, 2\pi)$.

Let $x(t)$ and $y(t)$ be real polynomials with respect to t, $t \in [0, 1]$. A planar curve $\boldsymbol{P}(t) = x(t) + \mathbf{i}y(t)$ is called a *Pythagorean-hodograph (PH)* curve if and only if its derivative $\boldsymbol{P}'(t) = x'(t) + \mathbf{i}y'(t)$ satisfies the Pythagorean condition

$$x'^2(t) + y'^2(t) = \sigma^2(t),$$

for some real polynomial $\sigma(t)$ [7]. Equivalently, a planar curve $\boldsymbol{P}(t)$ is a PH curve if and only if

$$\boldsymbol{P}'(t) = w(t)[u(t) + \mathbf{i}v(t)]^2, \tag{3}$$