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Approximation of Quasi-Monte Carlo worst case error in weighted spaces of infinitely times smooth functions

Makoto Matsumoto [a], Ryuichi Ohori [b], Takehito Yoshiki [c,*]

[a] Graduate School of Sciences, Hiroshima University, Hiroshima 739-8526, Japan
[b] Fujitsu Laboratories Ltd., Kanagawa 211-8588, Japan
[c] Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8561, Japan

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ABSTRACT

In this paper, we consider Quasi-Monte Carlo (QMC) worst case error of weighted smooth function classes in $C^\infty[0,1]^s$ by a digital net over $\mathbb{F}_2$. We show that the ratio of the worst case error to the QMC integration error of an exponential function is bounded above and below by constants. This result provides us with a simple interpretation that a digital net with small QMC integration error for an exponential function also gives the small integration error for any function in this function space.

1. Introduction

Quasi-Monte Carlo (QMC) integration is one of methods for numerical integration over the s-dimensional unit cube $[0,1)^s$ (see [1–3] for details). We approximate the integral of a function $f : [0,1)^s \to \mathbb{R}$

$$I(f) := \int_{[0,1)^s} f(\boldsymbol{x})\, d\boldsymbol{x}$$

by a quadrature rule of the form

$$I_P(f) := \frac{1}{N} \sum_{\boldsymbol{x} \in P} f(\boldsymbol{x}),$$

where P is a point set in $[0,1)^s$ with finite cardinality N. We define the (signed) integration error by $\mathrm{Err}\,(f; P) := I_P(f) - I(f)$. In order to make the absolute integration error $|\mathrm{Err}\,(f; P)|$ small for a class of functions, we often measure the quality of point sets $P \subset [0,1)^s$ using the so-called worst case error. For a function space F with norm $\|\cdot\|_F$, the worst case error for F by P is defined as the supremum of the absolute value of the integration error in the unit ball of F:

$$\mathrm{e}^{\mathrm{wor}}(F; P) := \sup_{\substack{f \in F \\ \|f\|_F \le 1}} |\mathrm{Err}\,(f; P)|\,. \tag{1}$$

Apparently, this quantity performs as an upper bound on $|\mathrm{Err}\,(f; P)|$ for every $f \in F$:

$$|\mathrm{Err}\,(f; P)| \le \mathrm{e}^{\mathrm{wor}}(F; P) \cdot \|f\|_F\,. \tag{2}$$

* Corresponding author.
E-mail addresses: m-mat@math.sci.hiroshima-u.ac.jp (M. Matsumoto), ohori.ryuichi@jp.fujitsu.com (R. Ohori), yoshiki.takehito.47x@st.kyoto-u.ac.jp (T. Yoshiki).

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Thus, our goal is to find a point set P with small value of $\mathrm{e}^{\mathrm{wor}}(F; P)$ for a given function class F. In what follows, we focus on QMC rules using digital nets over $\mathbb{F}_2$ as point sets (see Definition 2.1). This is a special type of construction scheme we often use for practical application of QMC.

In the previous works on QMC, the function class consisting of functions f whose derivatives up to order 1 are continuous has been extensively considered. For this function class, the star-discrepancy plays an alternative role to the worst case error in (2). Various types of low discrepancy digital nets P have been developed, whose discrepancy is of order $N^{-1+\epsilon}$ for arbitrary small $\epsilon > 0$ (see [1,4] for the details). Recently, smoother function spaces have also become targets of research in studies of QMC rules, e.g., α-smooth Sobolev space, consisting of functions whose mixed derivatives up to order α in each coordinate are continuous for $\alpha \geq 2$. For this function space, efficient digital nets P have also been established (see e.g., [5,6]). They satisfy the higher order convergence of the worst case error $N^{-\alpha+\epsilon}$.

More recently, for some family of functions in $C^\infty[0, 1]^s$, the existence or theoretical construction algorithm of digital nets P have been developed with the convergence rate of the integration error $N^{-C_s \log N}$ for a constant C_s depending on s [7,8]. On the other hand, heuristic algorithm is also used for constructing digital nets giving efficient QMC rules in [9], in which Walsh Figure of Merit (WAFOM) is of great importance. WAFOM performs as an upper bound on the worst case error of a digital net P. Since WAFOM is computable quickly on computers, we can obtain low-WAFOM point sets by computer search (see e.g., [9–11]).

As a continuous work, Suzuki [12] considered more general function spaces

$$\mathcal{F}_{s,\boldsymbol{u}} := \left\{ f \in C^\infty[0, 1]^s \;\middle|\; \|f\|_{\mathcal{F}_{s,\boldsymbol{u}}} := \sup_{(\alpha_1,\ldots,\alpha_s)\in(\mathbb{N}\cup\{0\})^s} \frac{\left\|f^{(\alpha_1,\ldots,\alpha_s)}\right\|_{L^1}}{\prod_{1\leq j\leq s} u_j^{\alpha_j}} < \infty \right\},$$

with a sequence of positive weights $\boldsymbol{u} = (u_j)_{j\geq 1}$. (Actually he considered another function space including this smooth function space.) Here $f^{(\alpha_1,\ldots,\alpha_s)}$ is the $(\alpha_1, \ldots, \alpha_s)$th mixed partial derivative of f. When we set $u_j = 2$ for all j, this space corresponds to the original space considered in the above works. Suzuki [12] gave the existence of digital nets which achieve the convergence rate $N^{-C_{s,\boldsymbol{u}} \log N}$ of the worst case error for a constant $C_{s,\boldsymbol{u}}$ depending on s and $\boldsymbol{u}$. Furthermore, under certain conditions on the weights $\boldsymbol{u}$, this upper bound on the worst case error becomes $C' \cdot N^{-D'(\log N)^{E'}}$ for absolute constants C', D', E', which is a dimension-independent convergence rate. Computer search algorithm is also effective for finding good point sets in this space as in the above case. For this function space, we can introduce a generalization of WAFOM as an upper bound on the worst case error (see [12, Remark 6.4]).

In this paper, we give feasible upper and lower bounds on the worst case error $\mathrm{e}^{\mathrm{wor}}\left(\mathcal{F}_{s,\boldsymbol{u}}; P\right)$ for the function space $\mathcal{F}_{s,\boldsymbol{u}}$ by using well-known exponential functions:

$$L_{s,\boldsymbol{u}} \leq \frac{\mathrm{Err}\left(\exp(-\sum_{1\leq j\leq s} u_j x_j); P\right)}{\mathrm{e}^{\mathrm{wor}}\left(\mathcal{F}_{s,\boldsymbol{u}}; P\right)} \leq U_{s,\boldsymbol{u}},$$

for any digital net P and some constants $L_{s,\boldsymbol{u}}$ and $U_{s,\boldsymbol{u}}$ depending on s and $\boldsymbol{u}$ but not on P (see Theorem 3.1 for the detailed description of the main theorem). Although this is not an equality for the worst case error and we restrict the range of P to the class of digital nets, this gives us a simple interpretation for the worst case error.

In the proof of the above inequalities, we use a figure of merit $W_{\boldsymbol{u}}(P)$ of P, which is a generalization of WAFOM (see Definition 3.3 for $W_{\boldsymbol{u}}(P)$) as mentioned above. It is proved that $W_{\boldsymbol{u}}(P)$ approximates $\mathrm{e}^{\mathrm{wor}}\left(\mathcal{F}_{s,\boldsymbol{u}}; P\right)$ (see Lemma 3.6) and bounds the worst case error:

$$L'_{s,\boldsymbol{u}} \leq \frac{W_{\boldsymbol{u}}(P)}{\mathrm{e}^{\mathrm{wor}}\left(\mathcal{F}_{s,\boldsymbol{u}}; P\right)} \leq U'_{s,\boldsymbol{u}},$$

where $L'_{s,\boldsymbol{u}}$ and $U'_{s,\boldsymbol{u}}$ depends not on P but on s and $\boldsymbol{u}$ (see Corollary 3.11 for the explicit statement). This result verifies that these two criteria have essentially the same role in the QMC error analysis. Since both $W_{\boldsymbol{u}}(P)$ and $\mathrm{Err}\left(\exp(-\sum_{1\leq j\leq s} u_j x_j); P\right)$ are computable criteria for the quality of digital nets, we can consider computer search for finding efficient QMC rules (see first three items of Remark 3.13).

The remainder of this article is organized as follows. In Section 2 we recall some definitions for QMC integration by digital nets, and a relation with Walsh coefficients. In Section 3 we prove the main result. In Section 4, we show the numerical properties on $\mathrm{Err}\left(\exp(-\sum_{1\leq j\leq s} u_j x_j); P\right)$ and $W_{\boldsymbol{u}}(P)$ in terms of the ratio of these two quantities.

2. Preliminaries

We denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in this paper. In this section we briefly recall the notion of digital nets and Walsh coefficients.

2.1. Digital nets

We first introduce digital nets over the two-element field $\mathbb{F}_2 = \{0, 1\}$, which are defined as follows.