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Multi-level Monte Carlo weak Galerkin method with nested meshes for stochastic Brinkman problem

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ABSTRACT

This paper is devoted to the numerical analysis of a multi-level Monte Carlo weak Galerkin (MLMCWG) approximation with nested meshes for solving stochastic Brinkman equations with two dimensional spatial domain. With weak gradient operator and a stabilizer at hand, the weak Galerkin (WG) technique is a high-order accurate and stable method which can easily handle deterministic partial differential equations with complex geometries, flows with jump fluid viscosity coefficients or high-contrast permeability fields given by each sample. The multi-level Monte Carlo (MLMC) technique with nested meshes balances the sampling error and the spatial approximation error, where the computational cost can be sharply reduced to log-linear complexity with respect to the degree of freedom in spatial direction. The nested meshes requirement is introduced here in order to simplify the analysis, which can be generalized to MLMC with non-nested meshes. Error estimates are derived in terms of the spatial meshsize and the number of samples. The numerical tests are provided to illustrate the behavior of the MLMCWG method and verify our theoretical results regarding optimal convergence of the approximate solutions.

1. Introduction

There are multitude of interesting models for the mathematical modeling of flows in porous media and materials, e.g., industrial filters, natural water management, dolomite or limestone oil reservoirs, etc.. In these applications, the Darcy model can handle the slow flow problems [1], while it does not predict the cavity problems well. The Stokes model is the steady state of the Navier–Stokes equation, which is capable of modeling flow of cavity problems [2]. The behavior of viscous fluid in cavity and porous media can be captured accurately by the Darcy–Stokes interface model provided the priori information of the interface is given. However, the location and the geometry of the interface are unattainable in many applications, and the interface jump conditions are difficult to impose experimentally even with precise information of the interface at hand. The Brinkman model of porous media, which is a generalization of the Stokes equation and an approximation of the Navier–Stokes equations at low Reynolds numbers, behaves like a Darcy flow and a Stokes flow for the regions with large and small permeability values, respectively [3].

The Brinkman model is a combination of Darcy's and Stokes' equations, which is a very effective model for flows in highly heterogeneous media in real applications. Therefore, the accuracy of the Brinkman flow simulation is of significant practical interest, but it is not easy to design uniform stable efficient algorithms to capture the behavior of the Darcy flow and the Stokes flow in different permeability regions simultaneously. Generally speaking, the usual Darcy stable method does not

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work well for the Stokes flow and vice versa. The Darcy stable elements, such as the Raviart–Thomas elements, although perform well in the regions with large permeability value, perform poorly in the Stokes regions [4]. On the other hand, the Stokes stable elements, such as the conforming P_2–P_0 element and the nonconforming Crouzeix–Raviart element, perform well in the regions with small permeability value and perform poorly in the Darcy regions [5]. Many algorithms are proposed by modifying either existing Darcy stable elements or Stokes stable elements to design Brinkman stable elements, which are uniformly stable in both Darcy's and Stoke's regions (see [5,6] and references therein). Recently, Mu, Wang, and Ye proposed a weak Galerkin scheme for the deterministic Brinkman equation, which is uniformly stable for large and small permeability regions, based on the weak gradient operator [7]. The most important advantage of this method is that the same formulation works well for both the Darcy and the Stokes problems.

In many practical applications, the interfaces of the Darcy and Stokes domain are unknown beforehand, which is equivalent to that the permeabilities are random variables or the fluid viscosities coefficients jumps stochastically. The stochastic partial differential equation (SPDE) is a powerful tool which adequately describes the behavior of flows in highly heterogeneous media with stochastic permeability and viscosity. Let $D \subset \mathbb{R}^2$ be a bounded convex domain with piecewise smooth boundary and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Here, we shall consider the following stochastic Brinkman equation: find the velocity $\mathbf{u}(x, \omega) : (\overline{D} \times \Omega)^2 \to \mathbb{R}^2$ and the pressure $p(x, \omega) : \overline{D} \times \Omega \to \mathbb{R}$ of the fluid, such that

$$-\mu(\mathbf{x}, \omega)\Delta \mathbf{u} + \nabla p + \mu(\mathbf{x}, \omega)\kappa^{-1}(\mathbf{x}, \omega)\mathbf{u} = \mathbf{f}(\mathbf{x}), \qquad in\ D \times \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \qquad in\ D \times \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{g}(\mathbf{x}), \qquad on\ \partial D, \tag{3}$$

where the viscosity μ is a stochastic function with jump and κ is the stochastic permeability tensor (will be specified in the next section). Here the forcing term $\mathbf{f} \in L^2(D)^2$ and the boundary data $\mathbf{g} \in H^{\frac{1}{2}}(\partial D)^2$ are deterministic functions with the compatibility condition

$$\int_{\partial D} \mathbf{g} \cdot \mathbf{n} ds = 0.$$

This type of stochastic problems have many applications in industrial and engineering phenomena, such as groundwater systems [8–10] and vuggy porous media [1,11]. For the sake of simplicity, we consider $\mathbf{g} = 0$ in the sequel.

In this work, we will employ an efficient MLMCWG method for solving stochastic Brinkman equations (1)–(3). There are several merits of our algorithm. As mentioned above, one of the main advantages of the stochastic Brinkman model is that it can capture the Stokes and Darcy type flow behavior depending on the value of κ without priori information of the interface, and the WG method is uniformly stable for Darcy's and Stoke's regions for each realization of (1)–(3). Based on weak derivatives [12–15], the WG method shall provide a systematic framework for dealing with high-oscillation or discontinuity of the solutions near the interface of Darcy's and Stokes' regions. Secondly, the MLMCWG method requires suboptimal computational cost of log-linear complexity, in terms of the degree of freedom used in the WG approximation. The multi-level Monte Carlo technique has been widely used to replace traditional MC-like methods, so that the computational cost can be sharply reduced [16–19]. Finally, the interfaces between Darcy's and Stokes' regions are always complicated, and the traditional triangular partitions may not be suitable for practical computations. On the other hand, the WG method allows arbitrary polygons as long as the partitions are shape regular (cf. [20]) which is more efficient than the standard finite element method (FEM), and the corresponding MLMCWG is superior than MLMCFEM for stochastic case. The convergence analysis of the WG method with different polygons are guaranteed under the same framework, which makes this method more flexible and robust.

The remainder of this paper is organized as follows. In Section 2, the preliminaries including relevant terminologies and notations are presented. In Section 3, we show the weak derivatives, variational formulations, and the weak Galerkin method for the stochastic Brinkman problem. This is followed by the description of the single level Monte Carlo weak Galerkin (SLMCWG) method and the multi level Monte Carlo weak Galerkin method for stochastic Brinkman equations (1)–(3) in Section 3. In Section 4, we present the convergence results of SLMCWG and MLMCWG, respectively. The corresponding computational complexities are also analyzed. In Section 5, several numerical simulations are provided to demonstrate the efficiency of our algorithms. The last section is devoted to some concluding remarks.

2. Preliminaries

2.1. Terminologies

A multi-index $\alpha = (\alpha_1, \alpha_2)$ is a two-tuple of non-negative integers with its length given by $|\alpha| = \alpha_1 + \alpha_2$. For a non-negative integer m, set

$$H^m(D) = \{v \in L^2(D); \partial^\alpha v \in L^2(D), 0 \le |\alpha| \le m\}$$