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Stabilization of low-order cross-grid P_kQ_l mixed finite elements

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ABSTRACT

In this paper we analyze a low-order family of mixed finite element methods for the numerical solution of the Stokes problem and a second order elliptic problem, in two space dimensions. In these schemes, the pressure is interpolated on a mesh of rectangular elements, while the velocity is approximated on a triangular mesh obtained by dividing each rectangle into four triangles by its diagonals. For the lowest order P_1Q_0, a global spurious pressure mode is shown to exist and so this element, as P_1Q_1 case analyzed in Armentano and Blasco (2010) , is unstable. However, following the ideas given in Bochev et al. (2006) , a simple stabilization procedure can be applied, when we approximate the solution of the Stokes problem, such that the new P_1Q_0 and P_1Q_1 methods are unconditionally stable, and achieve optimal accuracy with respect to solution regularity with simple and straightforward implementations. Moreover, we analyze the application of our P_1Q_1 element to the mixed formulation of the elliptic problem. In this case, by introducing the modified mixed weak form proposed in Brezzi et al. (1993) , optimal order of accuracy can be obtained with our stabilized P_1Q_1 elements. Numerical results are also presented, which confirm the existence of the spurious pressure mode for the P_1Q_0 element and the excellent stability and accuracy of the new stabilized methods.

1. Introduction

The approximation by mixed finite element methods of the Stokes problem has been widely developed. In some works the two independent variables, velocity and pressure, are approximated by using spaces of different order of approximation for each one [1–10]. On the other hand, some stabilized formulations, which consists in modifying the discrete problem by the addition of new terms which enhance its stability, are introduced in order to use the same order approximation spaces for the velocity and the pressure (see, for example, [8,9,11–19] and the references therein). In particular, standard C_0 finite element spaces of low polynomial orders remain a popular choice in many engineering applications because, besides their simplicity, they offer reasonable accuracy and uniform data structures when using equal order interpolation and so the develop of stabilization procedure is still a focus of the interest.

In [1] we introduce and analyze a new family of mixed finite element methods in which the pressure is interpolated on a mesh of rectangular elements and the velocity on a triangular mesh obtained by dividing each rectangle into four triangles by its diagonals. We denote by P_kQ_l the elements in which the velocity is interpolated in each triangle by polynomials of degree no greater than k and the pressure is interpolated in each rectangle by polynomials of degree in each variable no greater than l. In that work we proved the existence of a global spurious pressure mode for the P_1Q_1 element, and that the cross-grid P_2Q_1 element satisfies the inf–sup condition getting optimally convergent solutions.

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In the present work we analyze the lowest P_1Q_0 element, and we show the existence of a global spurious pressure mode, so that convergence of the pressure does not hold for this element. Then, following the ideas given by Bochev, Dohrmann and Gunzburger in [11], which we denote by [BDG] procedure, we present a stabilized finite element method for the Stokes problem to counteract the lack of stability for our P_1Q_0 and P_1Q_1 cross-grid elements. The goal of these stabilized methods is that, in contrast to other stabilization procedures, they are parameter free, always lead to symmetric linear systems and have simple and straightforward computational implementation.

On the other hand, in the mixed formulation of second order elliptic problems, the approximation of the two variables has to be done taking into account some particular compatibility conditions in order to avoid instabilities (see [2,20,21] and the references therein). Moreover, the stable finite element approximations could be different from this problem to the Stokes problem, for example, the mini-elements are stable for the Stokes problem but not for the elliptic (see, for example, [2,22]). In this work we also analyze the application of our P_1Q_1 element to the mixed formulation of the elliptic problem. It is well known that in Raviart–Thomas spaces [23] (one of most used approaches for the elliptic problem), typically stable velocity approximations are continuous only in the normal direction and the most frequent approach RT_0P_0 has also discontinuous pressure [21–23]. In [24] the authors propose different approaches to stabilize the P_1P_1 elements, since the same strategy applied to the Stokes problem cannot be applied directly to the elliptic. In this paper we use the modified mixed weak form for the elliptic problem introduced in [20], which allows us to employ finite element spaces similar to those built for the Stokes problem. In particular, we can successfully apply our stabilized P_1Q_1 cross-grid elements to this modified mixed problem in order to obtain optimal equal-order continuous approximations. It is important to point out that the ideas present in this paper, could be extended to other interesting problems (see, for example, [24–28] and the references therein), with the purpose to obtain optimal, continuous and economical approximations.

Some numerical results are also presented which confirm the presence of the spurious pressure mode for the P_1Q_0 element and the successful stabilization procedure for our P_1Q_0 and P_1Q_1 cross-grid elements for the Stokes problem. Moreover, we show a numerical example for the application of our stabilized P_1Q_1 element to the modified mixed problem associated to the elliptic, which shows the good performance of our approximation method. Although in the stabilized procedure provided here we consider only rectangular elements, the methods we have developed can also be applied to meshes of general quadrilateral elements.

The rest of the paper is organized as follows. In Section 2 we state the Stokes problem, introduce the P_kQ_l mixed finite element approximations and we prove the instability of P_1Q_0, the lowest order case. In Section 3 we present the stabilization procedure for the cross-grid elements P_1Q_0 and P_1Q_1. In Section 4 we show the modified mixed formulation for second order elliptic equations and the approximation properties utilizing the stabilization method introduced in Section 3. Finally, in Section 5 we present some numerical examples which show the good performance of the stabilization procedure.

2. Cross-grid P_kQ_l finite element approximation of the Stokes problem

In this section we recall the Stokes problem and the family of cross-grid P_kQ_l mixed finite element methods introduced in [1] for its numerical approximation.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and polygonal domain, the classical Stokes problem is given by: Find the fluid velocity $\mathbf{u}$ and the pressure p such that

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega\,, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\,, \\ \mathbf{u} = 0 & \text{on } \Gamma := \partial\Omega\,, \end{cases} \tag{2.1}$$

where $\mathbf{f} \in (H^{-1}(\Omega))^2$ (the dual space of $(H_0^1(\Omega))^2$) is a given body force per unit mass and $\mu > 0$ is the kinematic viscosity, which we assume constant.

Let $V := (H_0^1(\Omega))^2$ and $Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$. The weak form of (2.1) is given by: Find $\mathbf{u} \in V$ and $p \in Q$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \langle \mathbf{f},\mathbf{v}\rangle_{V'\times V} & \forall \mathbf{v} \in V\,, \\ b(\mathbf{u},q) = 0 & \forall q \in Q\,, \end{cases} \tag{2.2}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined on $V\times V$ and $V\times Q$, respectively, as

$$a(\mathbf{u},\mathbf{v}) = \mu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} \qquad \mathbf{u},\mathbf{v} \in V\,,$$

$$b(\mathbf{v},q) = -\int_\Omega \nabla\cdot\mathbf{v}\, q \qquad \mathbf{v} \in V, q \in Q\,.$$

We denote by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ the norms and seminorms in $H^m(D)$ or $(H^m(D))^2$ respectively and $(\cdot,\cdot)_D$ denotes the inner product in $L^2(D)$ or $(L^2(D))^2$ for any subdomain $D \subset \Omega$. The domain subscript is dropped for the case $D = \Omega$.

It is well known that the bilinear form $a(\cdot,\cdot)$ is coercive in V and there exists a constant $\beta > 0$ (see for instance [21]) such that for all $q \in Q$

$$\sup_{0\neq\mathbf{v}\in V} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1} \geq \beta\|q\|_0.$$