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Comparison of numerical methods for the Zakharov system in the subsonic limit regime

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ABSTRACT

We compare numerically spatial/temporal resolution of various methods for solving the Zakharov system (ZS) in the subsonic limit regime, which involves a small parameter $0 < \varepsilon \le 1$ inversely proportional to the acoustic speed. In this regime, i.e., $0 < \varepsilon \ll 1$, the solution presents highly oscillatory initial layers due to the wave operator or the incompatibility of the initial data. Specifically, the solution propagates waves with wavelength of $O(\varepsilon)$ and $O(1)$ in time and space, respectively. By applying the sine pseudospectral discretization for spatial derivatives followed by a time-splitting technique for integrating the Schrödinger equation combined with an exponential wave integrator in phase space for integrating the wave equation, we propose four different numerical methods for the ZS based on different quadrature rules for approximating the integral or some property of conservation. Numerical results suggest that all the methods are spectrally accurate in space, which is uniformly for $\varepsilon \in (0, 1]$. For temporal error, the best method converges uniformly with linear convergence rate at $O(\tau)$ in the subsonic limit regime.

1. Introduction

The Zakharov system (ZS), which plays an important role in plasma physics, was firstly derived by V. Zakharov [1] for describing the propagation of Langmuir waves in plasma

$$\begin{cases} i\partial_t E(\mathbf{x},t) + \Delta E(\mathbf{x},t) - N(\mathbf{x},t)E(\mathbf{x},t) = 0, & \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ \varepsilon^2\partial_{tt}N(\mathbf{x},t) - \Delta N(\mathbf{x},t) - \Delta|E(\mathbf{x},t)|^2 = 0, & \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ E(\mathbf{x},0)=E_0(\mathbf{x}), \quad N(\mathbf{x},0)=N_0(\mathbf{x}), \quad \partial_t N(\mathbf{x},0) = N_1(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{1.1}$$

Here t is time, $\mathbf{x}$ is the spatial variable; the complex, dispersive field $E := E(\mathbf{x}, t)$ represents a varying envelope of a highly oscillatory electric field; the real, nondispersive field $N := N(\mathbf{x}, t)$ is the fluctuation of the plasma ion density from its equilibrium value, and $0 < \varepsilon \le 1$ is a dimensionless parameter which is inversely proportional to the acoustic speed, and $E_0(\mathbf{x})$, $N_0(\mathbf{x})$ and $N_1(\mathbf{x})$ are given functions satisfying $\int_{\mathbb{R}^d} N_1(\mathbf{x})d\mathbf{x} = 0$. The ZS has applications in plasma physics [1,2], molecular chains [3], hydrodynamics [4,5] and so on. It is a universal model for the study of the interaction between a dispersive wave and a nondispersive wave.

As is known, the ZS (1.1) conserves the *mass* as

$$\mathcal{M}(t) =: \|E(\cdot,t)\|^2 = \int_{\mathbb{R}^d} |E(\mathbf{x},t)|^2 d\mathbf{x} \equiv \int_{\mathbb{R}^d} |E_0(\mathbf{x})|^2 d\mathbf{x} = \mathcal{M}(0), \quad t \ge 0, \tag{1.2}$$

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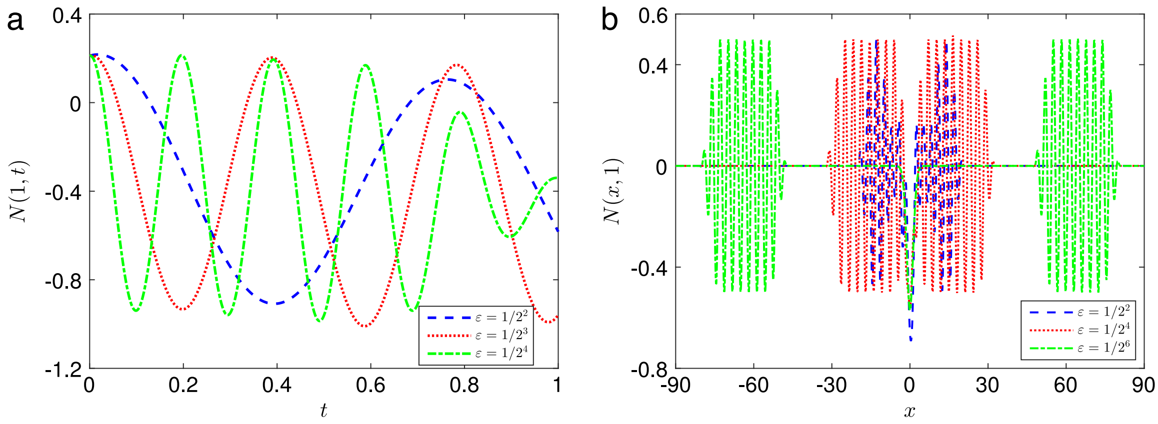

Fig. 1. Profiles of $N(x=1,t)$ (a) and $N(x,t=1)$ (b) of (1.1) with (1.6) and (1.7) for $d=1$ and different ε.

and the *energy* as

$$\mathcal{L}(t) := \int_{\mathbb{R}^d} \left[|\nabla E|^2 + N|E|^2 + \frac{1}{2}\left(|\nabla U|^2 + |N|^2\right) \right] d\mathbf{x} \equiv \mathcal{L}(0), \quad t \geq 0, \tag{1.3}$$

where $U := U(\mathbf{x},t)$ is defined as

$$-\Delta U(\mathbf{x},t) = \varepsilon \partial_t N(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^d; \quad \lim_{|\mathbf{x}|\to\infty} U(\mathbf{x},t) = 0, \quad t \geq 0. \tag{1.4}$$

For the ZS (1.1) with $\varepsilon = 1$, i.e., $O(1)$-speed of sound regime, there are extensive analytical and numerical results in the literatures. For the derivation of ZS from the Euler–Maxwell equations, we refer to [6,7]; for the well-posedness on $\mathbb{R}^d$, we refer to [6,8–11] and references therein; for the well-posedness in Sobolev spaces of lower regularity on $\mathbb{T}^d$, we refer to [12,13]; and for the extensions to the generalized Zakharov system and the vector Zakharov system, we refer to [14–16]. For the numerical part, many methods have been proposed for the ZS. For example, Payne et al. [17] presented a spectral method for ZS in 1D, where a truncated Fourier expansion was used to eliminate the aliasing errors. Glassey [18] designed an energy-preserving implicit finite difference scheme and proved its convergence at the first order in both spatial and temporal discretizations. The convergence rate was improved to the optimal second order in [19,20] by an implicit or a semi-explicit conservative finite difference scheme. Other approaches include the Jacobi-type method [21], exponential wave integrator spectral method [17,22], Legendre–Galerkin method [23], discontinuous-Galerkin method [24] and time-splitting spectral method [25,26].

However, for the ZS (1.1) with $0 < \varepsilon \ll 1$, i.e., in the subsonic limit regime, the analysis and efficient computation are mathematically rather complicated issues due to that the solution is highly oscillatory in time. There is extensive mathematical analysis of the subsonic limit of the ZS (1.1) to the cubic nonlinear Schrödinger (NLS) equation [27–29]

$$\begin{cases} i\partial_t E^s(\mathbf{x},t) + \Delta E^s(\mathbf{x},t) + |E^s(\mathbf{x},t)|^2 E^s(\mathbf{x},t) = 0, & \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ E^s(\mathbf{x},0) = E_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d. \end{cases} \tag{1.5}$$

Based on the rigorous analysis in [27–29], the solution of ZS (1.1) propagates highly oscillatory waves with wavelength $O(\varepsilon)$ and $O(1)$ in time and space, respectively, and rapid outgoing initial layers at speed $O(1/\varepsilon)$ in space, when $0 < \varepsilon \ll 1$. To illustrate this, Fig. 1 displays the solution of (1.1) for $d = 1$ and initial data

$$\begin{aligned} &E_0(x) = e^{-x^2/2 + i\pi/3}, \quad N_0(x) = -|E_0(x)|^2 + g\left(\frac{x+18}{10}\right) g\left(\frac{18-x}{9}\right) \sin\left(2x + \frac{\pi}{6}\right), \\ &N_1(x) = 2\,\mathrm{Im}\left[\overline{E_0(x)} E_0''(x)\right] + e^{-x^2/3} \sin(2x), \end{aligned} \tag{1.6}$$

with χ_Ω the characteristic function of the domain Ω and

$$g(x) = \frac{f(x)}{f(x) + f(1-x)}, \qquad f(x) = e^{-1/x} \chi_{(0,\infty)}, \qquad x \in \mathbb{R}. \tag{1.7}$$

This highly oscillatory nature of the solution of the ZS (1.1) in time brings significant numerical burdens, making the numerical approximation challenging and costly in the subsonic limit regime $0 < \varepsilon \ll 1$. Some numerical methods, including the finite difference method [30,31] and the time-splitting spectral method [25,26], have been proposed and analyzed for solving ZS in the subsonic limit regime. Based on their results, in order to obtain the 'correct' oscillatory solution,