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A meshing strategy for a quadratic iso-parametric FEM in cavitation computation in nonlinear elasticity ☆

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A B S T R A C T

The approximation properties of a quadratic iso-parametric finite element method for a typical cavitation problem in nonlinear elasticity in two dimensions are analyzed. More precisely, (1) the finite element interpolation errors are established in terms of the mesh parameters; (2) a mesh distribution strategy based on an error equi-distribution principle is given; (3) the convergence of finite element cavity solutions is proved. Numerical experiments show that, in fact, the optimal convergence rate can be achieved by the numerical cavity solutions.

1. Introduction

Nonlinear soft elastic materials, such as polymers, biological tissues, rubbers, etc., can display a particular singular deformation, referred to in the literature as cavitation, when strong external force is applied [1–5]. The occurrence and growth of cavities is considered closely related to the material instability and to the damage and failure mechanisms of the materials [6–10]. A huge amount of work has been done by numerous authors analyzing cavitation experimentally, theoretically as well as numerically.

Generally speaking, there are two representative approaches characterizing cavitation. One is the so-called defect model, which is based on the hypothesis that cavities grow from pre-existing micro defects. Under this assumption, Gent and Lindley [3] analyzed the critical hydrostatic pressure at which a given unit spherical void in an infinite extension of a Neo-Hookean material would blow up, which was in a good agreement with their experiments therein. The other is the perfect model established by Ball [11] based on the analytical evidence that, under certain circumstances, a deformation with cavities created in an originally intact material can be energetically favorable. It is shown that, under the assumption that the cavities can appear only at a finite number of fixed points in the intact materials, the solution of the defect model converges to the solution of the perfect model [12,13] as the radii of the pre-existing small voids go to zero. In addition, analytical

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and numerical evidences indicate that whether a point can serve as a possible position of cavitation can be evaluated by calculating the corresponding configurational forces [14,15].

The perfect model typically exhibits the Lavrentiev phenomenon [16] when there is a cavitation solution, leading to the failure of the conventional finite element methods [17,18]. Though there are existing numerical methods developed to deal with the Lavrentiev phenomenon [17–20], they do not seem to be suitable to tackle the cavitation problem. In fact, most of the numerical studies on cavitation are based on the defect model, in which one considers to minimize the total energy of the form

$$E(\boldsymbol{u}) = \int_{\Omega_\varrho} W(\nabla \boldsymbol{u}(\boldsymbol{x}))d\boldsymbol{x}, \tag{1.1}$$

in the set of admissible functions

$$\mathcal{A} = \{\boldsymbol{u} \in W^{1,p}(\Omega_\varrho; \mathbb{R}^n) \text{ is one-to-one a.e.} : \boldsymbol{u}|_{\Gamma_0} = \boldsymbol{u}_0, \det \nabla \boldsymbol{u} > 0 \text{ a.e.}\}, \tag{1.2}$$

where $\Omega_\varrho = \Omega \setminus \bigcup_{i=1}^m B_{\varrho_i}(\boldsymbol{a}_i) \subset \mathbb{R}^n$ $(n = 2, 3)$ denotes the region occupied by an elastic body in its reference configuration, $B_{\varrho_i}(\boldsymbol{a}_i) = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x} - \boldsymbol{a}_i| < \varrho_i\}$ is the pre-existing spherical hole centered at $\boldsymbol{a}_i$ with small radius $\varrho_i > 0$, $W : M_+^{n\times n} \to \mathbb{R}^+$ is the elastic stored energy density of the material, $M_+^{n\times n}$ denotes the set of $n \times n$ matrices with positive determinant, Γ_0 is the boundary of Ω.

A typical example of the elastic stored energy density is of the form

$$W(F) = \omega|F|^p + g(\det F), \quad \forall F \in M_+^{n\times n}, \tag{1.3}$$

where $\omega > 0$ is a material constant, $n - 1 < p < n$, and $g : (0, \infty) \to (0, \infty)$ is a continuously differentiable strictly convex function characterizing the compressibility of the material and satisfies

$$g(d) \to +\infty \text{ as } d \to 0, \text{ and } \frac{g(d)}{d} \to +\infty \text{ as } d \to +\infty. \tag{1.4}$$

As was shown by Ball [11], this kind of functional can have a singular minimizer displaying cavitation. Further studies on the existence of singular minimizers in Sobolev spaces are referred to [12,21,22].

One of the main difficulties in the computation of immense growth of voids is the orientation-preservation of the finite element deformation, which is a crucial constraint and is characterized by the pointwise positivity of the Jacobian determinant of the deformation gradient. For the conforming piecewise affine finite element, the condition leads to an unbearably large amount of degrees of freedom [23]. To overcome this difficulty, Lian and Li [24] proposed a dual-parametric finite element method for the symmetric cavitation problem and a quadratic iso-parametric finite element method for cavitation problems which allow multiple unsymmetrical prescribed defects of various shapes and sizes [25]. Xu and Henao [23] established a penalized non-conforming finite element method which is successfully applied to the computation of multiple cavities with sacrifice on the approximation accuracy near the cavities surface. However, strict analytical results are insufficient. The only practical analytical results for the cavitation computation known to the authors so far are [26], where a sufficient orientation-preservation condition and the interpolation error estimates were given for a dual-parametric bi-quadratic finite element method, and [27], where a set of sufficient and necessary orientation-preservation conditions for the quadratic iso-parametric finite element interpolation functions of radially symmetric cavity deformations are derived.

In this paper, we analyze the approximation properties of a quadratic iso-parametric finite element for the typical cavitation problem. The analytical results on the errors of finite element interpolation functions lead to a delicate relationship between the elastic energy error and the mesh parameters, which together with the orientation-preservation conditions (see Remark 3.4 and [27]) enable us to establish a mesh distribution strategy guaranteeing that the corresponding finite element cavitation solution is orientation preserving and its relative error on the elastic energy is $O(h^2)$, where h is the mesh size in the far field, *i.e.* a given distance away from the cavity. Above all, for the first time to our knowledge, the convergence of the finite element cavitation solutions in $W^{1,p}$ norm is proved, inspired by the recent important results on the convergence of energy minimizing sequences in nonlinear elasticity [28,29]. In fact, the numerical experiments show that the optimal order of convergence rate is achieved by the numerical cavitation solutions obtained on the meshes produced by our meshing strategy.

Since the cavitation solution is generally considered to be quite regular except in a neighborhood of the voids, where the material experiences extremely large expansion dominant deformations and the difficulty of the computation as well as analysis lies, we restrict ourselves to a simplified problem with $\Omega_\varrho = B_1(\mathbf{0}) \setminus B_\varrho(\mathbf{0})$ in $\mathbb{R}^2$ and a simple expansionary boundary condition given by $\boldsymbol{u}_0 = \lambda\boldsymbol{x}$. The results however can be extended to more general cases with much more tedious calculations, which we do not regard as a main concern of the present paper (see also Remark 4.4).

The structure of the paper is as follows. In Section 2, we introduce the iso-parametric finite element method and a radially symmetric large expansion accommodating triangulation method briefly. Section 3 is devoted to analyzing the interpolation errors of the cavitation solutions. The meshing strategy is given in Section 4, where the convergence theorem is also established. The numerical results are presented in Section 5. Some concluding remarks are made in Section 6.