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A multilevel decoupling method for the Navier–Stokes/Darcy model

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ABSTRACT

This paper considers a multilevel decoupling method for the coupled Navier–Stokes/Darcy model describing a free flowing fluid over a porous medium. The method utilizes a sequence of meshes on which a low dimensional fully coupled nonlinear problem is solved only on a very coarse initial mesh. On subsequent finer meshes, the approximate solution in each flow region is obtained by solving a linear decoupled problem and performing a correction step. The correction step in each domain is achieved by solving a linear system that differs from the original decoupled system only in the right hand side. We prove optimal error estimates and demonstrate that for a sequence of meshes with spacing $h_j = h_{j-1}^2$, the decoupling method is computationally efficient and achieves the same order of approximation as the fully coupled method.

1. Introduction

We consider a multilevel decoupling method for the Navier–Stokes/Darcy model. This model has a wide range of applications in science and engineering in scenarios where a free flowing fluid moves over a porous medium. This coupled problem has been studied extensively in the literature; see for example [1–10] and references therein. We mention [5] for an overview of results on the coupled model for approximations based on continuous finite elements (CG), [7,4] for numerical schemes based on discontinuous Galerkin finite elements (DG) and [6] for a multi-numerics scheme combining CG and DG methods in the free flow and porous media domains, respectively.

The finite element discretization of the fully coupled Navier–Stokes/Darcy model leads to a large, sparse, nonlinear and ill-conditioned algebraic system. Assembling and solving this non-linear system on large domains is computationally expensive; therefore, the development of efficient decoupling techniques is important not only for this problem, but also for other multi-physics couplings that may have the same general form. Indeed, the numerical analysis of coupled problems continues to garner interest in the direction of advancing computational models to be more sophisticated and physically relevant. A few examples include coupled free flow with multiphase flow, coupled dual porosity with free flow modeling flow in shale oil reservoirs, multiscale flow in severe regimes and fluid flow interacting with poroelastic material. We refer the reader to [11–17] for details on these topics.

The naturally decoupled structure of free flow and porous media flow domains means that the coupled problem lends itself well to numerical techniques that decouple the large nonlinear problem into two smaller subproblems in the respective subdomains. Domain decomposition methods for coupled (Navier–Stokes or Stokes)/Darcy models have been considered for example in [18–25]. This paper focuses on the numerical analysis and implementation of a three step multiple mesh decoupling method. This technique requires the solution of a small nonlinear coupled problem only on a very coarse mesh,

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then on subsequent finer meshes (up to a desired mesh size), the Navier–Stokes/Darcy model is decoupled into relatively smaller subproblems in each domain. The numerical scheme considered in this work combines the continuous finite element method in the free flow region and the DG method in the porous medium. This choice is motivated by the fact that the regular finite element method is adequate for the free flow regimes considered and DG method is numerically well suited to handle discontinuities that may arise in the porous medium [26]. The DG method also allows for easy implementation of high order approximations and satisfies local mass balance which is an important property for numerical approximation of flow problems in the porous medium.

This multilevel decoupling has been applied to the Stokes/Darcy model in [27] using continuous finite element methods. This method is a natural extension of the two-grid decoupling method considered for example in [28–33] for the Stokes/Darcy problem and in [34–37,17,38,10] for the Navier–Stokes/Darcy problem. In the two-grid decoupling method, the fully coupled problem is solved on a coarse grid of size h_0, then on a fine grid of size $h_1 = h_0^2$ or $h_1 = h_0^3$ recently in [36], the problem is decoupled into two smaller subproblems. The decoupling is achieved in one of two ways. A parallel approach in which the solution to the fully coupled problem on the coarse mesh as boundary data on the interface for each decoupled problem on the fine mesh. A sequential approach in which the Darcy problem is decoupled using the coarse mesh free flow velocity and the Darcy pressure on the fine mesh is used as a boundary condition for the decoupled free flow problem.

In this paper we consider a three step multilevel sequential decoupling scheme. The method starts with the solution of a small nonlinear coupled problem on a coarse mesh, then on a sequence of finer meshes, the problem is decoupled into two smaller subproblems. The coarse mesh free flow velocity is used as boundary data on the interface for the porous media flow problem. The resulting solution to the decoupled Darcy problem is then applied as boundary data on the interface for a modified linearized Navier–Stokes problem in the free flow region. In the third stage, the decoupled solutions are corrected on the fine mesh by solving linear systems that differ from the original decoupled problems only in the right hand side. The fine mesh correction step improves the quality of the numerical solution in comparison to the widely studied two-grid method. This correction has been applied to solve the Navier–Stokes problem in [38] and has recently been applied to the two-grid decoupling method for the Navier–Stokes/Darcy problem in [36].

The use of a sequence of intermediate finer meshes in the multilevel method allows for a very coarse initial mesh which means that one needs to solve a smaller nonlinear problem compared to the two-grid method. Further, since the DG method is used to approximate the solution in the porous medium, the resulting linear systems are larger compared to the continuous finite element method therefore the development of efficient decoupling strategies is of interest.

Our numerical experiments demonstrate that this multi-mesh decoupling scheme can result in significant computational savings for large problems. In addition, this technique has the potential to be extended to adaptive mesh refinement techniques between mesh levels. Multilevel finite element methods have been widely used in the literature; see for example [39–44]. In this paper we extend the analysis and implementation of the decoupled multilevel method in [27] to the nonlinear Navier–Stokes/Darcy case with a fine mesh correction. We perform a numerical comparison of the multilevel method to the fully coupled method in terms of accuracy and CPU times.

The paper is organized as follows. The fully coupled model and the corresponding finite element discretization are introduced in Sections 2 and 3, respectively. We introduce the multilevel finite element method and prove the convergence of the method in Sections 4 and 5. In Section 6 we provide numerical examples to demonstrate the convergence, robustness with respect to physical parameters and effectiveness in comparison to the fully coupled method. Conclusions follow.

2. Coupled Navier–Stokes/Darcy model

Let $\Omega \in \mathbb{R}^2$ be a bounded polygonal domain partitioned into two non-overlapping subdomains Ω_1 and Ω_2; for the free flow and porous media flow regions, respectively. The subdomains Ω_1 and Ω_2 are separated by a polygonal interface Γ_{12}. We denote the boundary of the free flow region by $\Gamma_1 = \partial\Omega \cap \partial\Omega_1$. The boundary of the porous medium ($\Gamma_2 = \partial\Omega \cap \partial\Omega_2$) is partitioned into disjoint sets $\Gamma_{2\mathrm{D}}$ and $\Gamma_{2\mathrm{N}}$, the Dirichlet and Neumann boundary edges, respectively, with the condition $|\Gamma_{2\mathrm{D}}| > 0$. We recall the equations governing the flow in each domain. The flow in Ω_1 is described by the Navier–Stokes equations

$$-\nabla \cdot (2\nu \boldsymbol{D}(\boldsymbol{u}) - p\boldsymbol{I}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{f}_1, \quad \text{in } \Omega_1, \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega_1, \tag{2.1b}$$

$$\boldsymbol{u} = 0, \quad \text{on } \Gamma_1. \tag{2.1c}$$

The variables $\boldsymbol{u}$ and p denote the Navier–Stokes velocity and pressure, respectively. The coefficient ν is the kinematic viscosity of the fluid, the function $\boldsymbol{f}_1$ is the external force acting on the free fluid and $\boldsymbol{D}(\boldsymbol{u})$ is the rate of strain matrix

$$\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \boldsymbol{u}^T).$$

The flow in the porous medium is governed by Darcy's Law

$$-\nabla \cdot \boldsymbol{K}\nabla\phi = f_2, \quad \text{in } \Omega_2, \tag{2.2a}$$

$$\phi = 0, \quad \text{on } \Gamma_{2\mathrm{D}}, \tag{2.2b}$$

$$\boldsymbol{K}\nabla\phi \cdot \boldsymbol{n}_2 = g_{\mathrm{N}}, \quad \text{on } \Gamma_{2\mathrm{N}}. \tag{2.2c}$$