Journal of Computational and Applied Mathematics 325 (2017) 113–124

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Conservative integrators for a toy model of weak turbulence

Aquil D. Jones, Gideon Simpson *, William Wilson

Department of Mathematics, Drexel University, Philadelphia, PA, USA

ARTICLE INFO

Article history:
Received 1 July 2016
Received in revised form 9 April 2017

MSC:
35Q55
34A33
65P10
65L20

Keywords:
Nonlinear Schrödinger
Weak turbulence
Conservative integrators

ABSTRACT

Weak turbulence is a phenomenon by which a system generically transfers energy from low to high wave numbers, while persisting for all finite time. It has been conjectured by Bourgain that the 2D defocusing nonlinear Schrödinger equation (NLS) on the torus has this dynamic, and several analytical and numerical studies have worked towards addressing this point.

In the process of studying the conjecture, Colliander, Keel, Staffilani, Takaoka, and Tao introduced a "toy model" dynamical system as an approximation of NLS, which has been subsequently studied numerically. In this work, we formulate and examine several numerical schemes for integrating this model equation. The model has two invariants, and our schemes aim to conserve at least one of them. We prove convergence in some cases, and our numerical studies show that the schemes compare favorably to others, such as Trapezoidal Rule and fixed step fourth order Runge–Kutta. The preservation of the invariants is particularly important in the study of weak turbulence as the energy transfer tends to occur on long time scales.

1. Introduction

In [1], the 2D, defocusing, cubic, toroidal nonlinear Schrödinger equation (NLS),

$$iu_t + \Delta u - |u|^2 u = 0, \qquad u(0,x) = u_0(x) \quad \text{for } x \in \mathbb{T}^2, \tag{1.1}$$

was approximated by a "Toy Model System" given by the equation

$$-i\dot{b}_j = -\left|b_j\right|^2 b_j + 2b_{j-1}^2 \bar{b}_j + 2b_{j+1}^2 \bar{b}_j, \quad \text{for } j = 1 \dots N \tag{1.2}$$

with boundary conditions

$$b_0(t) = b_{N+1}(t) = 0. \tag{1.3}$$

Subject to these boundary conditions, (1.2) conserves ℓ^2,

$$\mathcal{M}[\mathbf{b}(t)] = \sum_{j=1}^{N} |b_j(t)|^2, \tag{1.4}$$

* Corresponding author.
E-mail address: simpson@math.dexel.edu (G. Simpson).

http://dx.doi.org/10.1016/j.cam.2017.04.038

and the Hamiltonian

$$\mathscr{H}[\mathbf{b}(t)] = \sum_{j=1}^{N}\left(\frac{1}{4}|b_j(t)|^4 - \mathrm{Re}(\bar{b}_j(t)^2 b_{j-1}(t)^2)\right). \tag{1.5}$$

Indeed, the flow of (1.2) can be expressed as

$$i\dot{b}_j = 2\frac{\partial \mathscr{H}[\mathbf{b}]}{\partial \bar{b}_j}. \tag{1.6}$$

In this way, we can interpret (1.2) as a Hamiltonian system of nonlinearly and degenerately coupled oscillators.

1.1. Weak turbulence

Roughly, $|b_j(t)|^2$ measures the spectral energy of u, the solution to (1.1), on a set of wave numbers, Λ_j. The sets Λ_j are tailored to have the property that the largest wave number in Λ_{j+1} exceeds the largest wave number in Λ_j. Thus, larger values of $|b_j|^2$ at larger values of j correspond to more energy of u in higher wave numbers.

The motivation for the approximation of (1.1) by (1.2) was to explore a long-standing hypothesis of Bourgain, that (1.1) could capture the phenomenon of weak turbulence, [2]. Loosely speaking, a weakly turbulent system exists globally in time, yet continuously propagates energy to ever higher frequencies. Thus, the norms tend to infinity, but are finite at all finite times. Another model equation for weak turbulence was formulated by Majda, McLaughlin, and Tabak, [3–8].

In [1], the authors proved that, given N, they could construct a solution of (1.2) which would propagate energy from $|b_j|^2$ at low index to high index j. This corresponds to a transfer of energy in (1.1), and, subject to rigorous analysis of the approximation, demonstrated that such energy cascades were present. However, this did not show that energy transfer in (1.1) was a generic phenomenon, an essential feature of weak turbulence. Analysis of this problem has continued in the recent works [9,10]. Separately, in [11,12], (1.2) was numerically simulated and observed to have such energy transfers for a variety of initial conditions for the lifespan of the simulations.

1.2. Relation to previous work

In [11,12], (1.2) was integrated using high order, adaptive, Runge–Kutta (RK) integrators. While the RK integrators gave high quality results for the duration of the simulations, they are unable to exactly conserve either of the two invariants.[1] At the same time, the energy transfer in the toy model system is slow, requiring integration out to long times. Thus, the RK integrators require significant computational effort to observe weak turbulence—small steps are needed for accuracy, but the phenomenological time of integration is long.

Given the interest in (1.2), the goal of this work is to present conservative methods that may aid in statistical studies of weak turbulence and other long time integration problems. The methods presented here are second order in the time step, Δt. In numerical experiments, we observe that comparatively large time steps can be taken with these schemes for exploring weak turbulence. While pointwise accuracy is lost with large Δt, the average energy transfer appears robust. In contrast, fixed time step RK schemes will, eventually, cease to provide accurate output.

A variety of strategies have been proposed for integrating Hamiltonian systems so as to preserve the invariants of the equations, including splitting methods, projection methods, symplectic methods, and the average vector field method, [13–15]. In this work, we explore some conservative discretizations of (1.2) which preserve either (1.4) or (1.5), including implicit midpoint. Some appear to be *ad hoc* and not from one of the aforementioned known discretization strategies, instead taking their inspiration from known discretizations of NLS, [16,17]. We also direct the reader to the recent work in [18], where the author tested an explicit symplectic integration scheme on (1.2) with a small number of nodes, $N = 5$.

1.3. Outline

Our paper is organized as follows. In Section 2, we formulate out schemes. Next, in Section 3, we prove a number of results on the integrators. Numerical simulations are presented in Section 4, and we discuss our results in Section 5.

2. Conservative integrators

In this section, we formulate our discretizations and prove that they conserve the relevant invariant, under a solvability assumption. These schemes are all symmetric, but the nonlinear terms are treated differently in each case. Since the dependent variables of (1.2) are nonlinearly and degenerately coupled, formulating a conservative scheme is nontrivial. This is contrast to NLS, where the spatial coupling, the analog of the lattice site coupling of (1.2), is linear.

[1] Conservation is only up to floating point error, which we do not consider here.