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The adaptive composite trapezoidal rule for Hadamard finite-part integrals on an interval

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ABSTRACT

In this article, we discuss an adaptive strategy of implementing trapezoidal rule for evaluating Hadamard finite-part integrals with kernels having different singularity. The purpose is to demonstrate cost savings and fast convergence rates engendered through adaptivity for the computation of finite-part integrals. The error indicators obtained from the *a posteriori* error estimate are used for mesh refinement. Numerical experiments demonstrate that the *a posteriori* error estimate is efficient, and there is no reliability-efficient gap.

1. Introduction

Consider the finite-part integral (see, e.g., [1–3])

$$\mathcal{I}u(y;s) = \fint_a^b \frac{u(x)}{|x-y|^{1+2s}}dx, \quad s \in (0,1) \tag{1}$$

with some arbitrary, but fixed $y \in (a,b)$, where $\fint$ denotes an integral in the Hadamard finite-part sense:

$$\fint_a^b \frac{u(x)}{|x-y|^{1+2s}}dx = \lim_{\epsilon\to 0}\left(\int_{\Omega_\epsilon} \frac{u(x)}{|x-y|^{1+2s}}dx - \frac{u(y)}{s\epsilon^{2s}}\right), \tag{2}$$

where $\Omega_\epsilon = (a,b)\setminus(y-\epsilon, y+\epsilon)$. The function $u(x)$ is said to be finite-part integrable with respect to the weight $|x-y|^{-1-2s}$ if the limit on the right hand side of (2) exists.

Integrals of this kind appear in many practical problems related to aerodynamics, wave propagation or fluid mechanics, mostly with relation to boundary element methods(BEMs) and finite-part integral equations [2]. Numerous work has

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been devoted in developing the efficient numerical evaluation method, such as Gaussian method [4,5], Newton–Cotes method [6,1,7–9], and some other methods [10–12]. Amongst them, Newton–Cotes rule is a popular one due to its ease of implementation and flexibility of mesh.

Error analysis of Newton–Cotes rule for Riemann integrals has been well done. The accuracy of Newton–Cotes rule with kth order piecewise polynomial interpolant for the usual Riemann integrals is $O(h^{k+1})$ for odd k and $O(h^{k+2})$ for even k. However, the rule is less accurate for finite-part integral (1) due to the hypersingularity of the kernel. For example, the correspondent result for finite-part integral with first-order singularity ($s = 0$) [13,14] and second-order singularity ($s = \frac{1}{2}$) [1,7,3] is only $O(h^k)$. The superconvergence of composite Newton–Cotes rule for finite-part integral (1) with second-order singularity was investigated in [6,15,3,16], where the higher-order accuracy can be reached on the condition that the singular point coincides with some *a priori* known point. Nevertheless, adaptivity, which is the topic here, is not covered in the above references. The key points in the design of adaptive quadrature rules are, first of all, to keep the number of function evaluations low, and secondly, to divide the domain of integration in such a way that the features of the integrand function are appropriately and effectively accounted for.

We analyze an h-adaptive Newton–Cotes rule for Hadamard finite-part integrals of the form

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}; \tag{3}$$

see Section 4 for a precise statement. In the context of the finite element method on shape-regular meshes, h-adaptive algorithms of this type (AFEM) have been analyzed in the last 20 years and are by now fairly well understood [17–22]. The situation is considerably less developed for Newton–Cotes rule for hypersingular integrals [23]. While several *a posteriori* error estimators for Riemann integrals are available in the literature (see [24–26] and the references therein), and numerous numerical studies indicate superiority of adaptive algorithms, an h-adaptive Newton–Cotes rule for Hadamard finite-part integrals still appears to be missing. Such an analysis is the main topic of the present paper.

In this paper, we construct the first-order composite Newton–Cotes rule (trapezoidal rule) for Hadamard finite-part integrals of the form (1), establish *a posteriori* error estimators of residual-type for different singular integral kernels, and design an h-adaptive algorithm based on these estimators. Finally, by means of a series of numerical experiments, we demonstrate that the proposed adaptive quadrature is capable of generating highly accurate approximations at a very low computational cost.

The rest of the paper is organized as follows. Section 2 introduces the precise notation of general (composite) trapezoidal rule for Hadamard finite-part integrals (1). Section 3 presents *a priori* and *a posteriori* error estimates analysis. Section 4 gives details of adaptive algorithms of the trapezoidal rule. Some numerical experiments demonstrate the efficiency of the *a posteriori* error estimates in Section 5.

2. Construction of the composite trapezoidal rule for (1)

As mentioned before, many researchers has made a lot of contributions to composite Newton–Cotes rule for finite-part integrals (see [6,1,7,8]), but all of these works are limited to the situation $s = \frac{1}{2}$. Until recently, a nodal-type trapezoidal rule is developed for (1) with $s \in [0, 1)$ in [9], where the singular point y is always chosen to be located at certain nodal point. In this section, we will derive the (composite) trapezoidal rule for (1) in the case that y is always located in a certain element.

Let

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

be the partition of $[a, b]$ with $h_i = x_{i+1} - x_i$ being the length of the element $e_i = (x_i, x_{i+1})$, $i = 0, 1, \ldots, n-1$. Denote $h = \max_{0 \le i \le n-1} h_i$ the size of the partition. The notation $A \lesssim B$ abbreviates $A \le C \cdot B$ with some generic constant $0 \le C < \infty$, which does not depend on h. We also assume that $y \in e_m$ for certain m satisfying $0 \le m \le n-1$.

Denote the piecewise linear Lagrange interpolant of $u(x)$ by

$$\pi_h u(x) = \sum_{i=0}^{n} u(x_i)\varphi_i(x), \tag{4}$$

where $\varphi_i(x)$ is the piecewise linear hat function, i.e.,

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_{i-1}}, & x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h_i}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1, \ldots, n-1,$$

and

$$\varphi_0(x) = \begin{cases} \dfrac{x_1 - x}{h_0}, & x \in [x_0, x_1], \\ 0, & \text{otherwise}, \end{cases} \qquad \varphi_n(x) = \begin{cases} \dfrac{x - x_{n-1}}{h_{n-1}}, & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}. \end{cases}$$