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A mass conservative scheme for solving the Vlasov–Poisson equation using characteristic curve☆

Dokkyun Yi [a], Sunyoung Bu [b,*]

[a] Division of Creative Integrated General Studies, Daegu University, KyungsangBukdo, 712-714, Republic of Korea

[b] Department of Liberal Arts, Hongik University, Sejong, 30016, Republic of Korea

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ABSTRACT

In this paper, we introduce a mass conservative scheme for solving the Vlasov–Poisson equation. This scheme is based on an Eulerian approach and is constructed using an interpolation scheme with limiters. In order to preserve the mass, the difference in the values for numerical flux functions on each cell is used; for this, the flux functions are constructed by preserving both the solution along a characteristics and the mass in each cell. We mainly investigate the conservation of L_1 and L_2 norms of the distribution function, total energy, entropy, and minimum value. In addition, we show that this scheme is bounded on the total variation. To demonstrate the efficiency of the proposed scheme, this scheme is compared with the flux balance scheme, Positive and Flux Conservative scheme, Umeda's scheme, and fifth order WENO reconstruction finite volume scheme.

1. Introduction

The Vlasov equation describes the evolution of the distribution function in phase space in collision-less plasmas under the effects of self-consistent electromagnetic fields. For this purpose, we focus on the numerical scheme to preserve positivity among many numerical techniques [1–10] for the Vlasov equation. A well-known scheme for preserving the positivity is Particle schemes (Lagrangian schemes or Particle-In-Cell scheme). In most cases, the numerical solution of the Vlasov equation is usually performed by Particle-In-Cell (PIC) schemes, which are based on tracing the motion of a finite number of macro-particles [11–14]. One of the advantages of these schemes is their computational economy, which enables easy application in high dimensional problems. However, a limitation on the number of particles may result in the appearance of numerical noise. To overcome this drawback, several schemes for discretizing the Vlasov equation on a mesh in phase space have been proposed in the Eulerian approach [15–21]. Compared to the Particle approach, a number of Eulerian approaches are superior in reducing numerical noise, which leads to better representation in low density regions of phase space. In addition, achieving high order accuracy for space discretization is another advantage of the Eulerian approach. However, high order accurate schemes sometimes generate numerical spurious oscillations, which are induced from non-preserving positivity. Moreover, they do not ensure that a numerical solution converges to a weak solution of the Vlasov equation.

Another challenge is the conservation of physical quantities (mass of the distribution function, total energy, entropy, and minimum value) in Vlasov equation solvers. For mass conservation, one of the finite volume type schemes, similar to

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* Corresponding author.
E-mail addresses: dkyi@daegu.ac.kr (D. Yi), syboo@hongik.ac.kr (S. Bu).

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the Eulerian approaches, is the flux balance scheme (FBM) [16], where discrete unknowns are averages of the distribution function on volumes paving the phase space. These unknowns are updated by considering incoming and outgoing fluxes leading to mass conservation. One of the common flaws of the flux balance scheme is non-preservation of positivity, which is inconvenient for lengthy simulations since numerical spurious oscillations occur as a result. As an alternative approach, a high-order conservative scheme called the Positive and Flux Conservative (PFC) scheme was developed [18]. Although the PFC scheme appears to be efficient for Vlasov simulations of several classical problems of plasma physics, numerical diffusion tends to be higher than other schemes. Umeda's scheme is also introduced [21] along with conservative schemes, which provide better characteristics for energy conservation. However, this scheme does not preserve positivity. Meanwhile, the essentially non-oscillatory (ENO) scheme [22,23] and the weighted essentially non-oscillatory (WENO) scheme [24] are introduced to obtain high-order and non-oscillatory schemes. To compare the numerical behavior of the WENO scheme for the Vlasov–Poisson (VP) system, the fifth order WENO reconstruction finite volume scheme is used for Vlasov–Poisson simulation.

The objective of this study is to propose a new scheme that provides a good approximation of the distribution function, the conservation of physical quantities (mass of the distribution function, total energy, entropy, and minimum value), and the preservation of positivity for the VP system. The purpose of this paper is threefold. The first is to introduce a detailed numerical procedure for the FBM, PFC, Umeda's scheme, and proposed scheme. The second is to compare the proposed scheme with other conservative schemes by performing one-dimensional Vlasov simulations to show that the proposed scheme enables better approximation of distribution than others and preserves positivity. The third is to show total variational stability of the proposed scheme for checking convergence. Several numerical schemes show that the proposed scheme is constructed with limiters for the stated objective and satisfies total variation stable (TV-stable), which guarantees a scheme that converges to the weak solution. Moreover, it is a higher-order scheme.

The remainder of this paper is organized as follows. In Section 2, we review the Vlasov–Poisson system and the existing schemes — FBM, PFC, and Umeda. In addition, the proposed scheme is described. In Section 3, the analysis of the proposed scheme and limiters are provided. In Section 4, numerical results are compared with the FBM, PFC, Umeda's, and fifth order WENO reconstruction finite volume schemes. Finally we present our conclusions in Section 5.

2. Preparation and concept

2.1. Vlasov–Poisson system

The motion of charged particles for collision-less plasmas is described by the well-known VP system,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = 0,$$

and

$$-\Delta_x \Phi = \rho, \qquad -\nabla_x \Phi = F,$$

where x and v are coordinates in phase space $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$, $d = 1, \ldots, 3$, F is the electric field, Φ is the self-consistent electrostatic potential, and $f(t, x, v)$ is the density distribution function, which represents the probability density of particles with velocity v at position x at time t. ρ is the charge density in the electrostatic case, coupled with the following distribution function:

$$\rho = \int f dv - 1.$$

Note that the model equations assume that the ions form an immobile charged background. In addition, the space and time units of the given equation are the Debye length and the inverse plasma frequency, respectively. To simplify the presentation of the scheme, the 1-dimensional electrostatic case is presented as follows:

$$\frac{\partial f}{\partial t} + v\partial_x f + F\partial_v f = 0, \tag{1}$$

$$-\partial_x^2 \Phi = \rho = \int_{-\infty}^{\infty} f dv - 1, \tag{2}$$

$$-\partial_x \Phi = F. \tag{3}$$

A periodic plasma of period L is considered. Hence, the functions f and F satisfy the periodic boundary conditions, as follows:

$$f(t, 0, v) = f(t, L, v), \quad v \in R,\ 0 \le t,$$
$$F(t, 0) = F(t, L), \quad 0 \le t.$$