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Convergence of IPDG for coupled time-dependent Navier–Stokes and Darcy equations

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ABSTRACT

A numerical method is proposed and analyzed for the coupled time-dependent Navier–Stokes equations and Darcy equations. Existence and uniqueness of the solution are obtained under a small data condition. A priori error estimates are derived. Numerical examples confirm the theoretical convergence rates.

1. Introduction

The coupling of free flow and porous media is an important model problem arising from several fields, such as energy, environment and bio-medicine. While the case of coupled Stokes and Darcy equations has been extensively studied mathematically and numerically (see a non-exhaustive list in the introduction of [1]), the case of coupled Navier–Stokes and Darcy equations has been the subject of very few publications. We refer to [2] for the steady-state coupling; both existence and uniqueness of a weak solution are proved and a priori error estimates for a numerical error are derived. Other related works are [3–7].

This article is dedicated to the numerical analysis for a discretization of the time-dependent Navier–Stokes equations coupled with Darcy equations. The interface conditions are the Beavers–Joseph–Saffman conditions. To our knowledge, the only previous publications on the time-dependent coupling are [8,9]; however the interface conditions in those two papers include inertial effects, which makes the analysis easier both for the weak solution and the numerical solution. In a more recent work, we showed existence and uniqueness of a weak solution to the coupled time-dependent Navier–Stokes and Darcy problem [1]. The present work is a continuation of this analysis, in the sense that we propose a discrete solution and we show optimal a priori error estimates between the weak solution and the numerical solution. We choose to discretize the equations by locally mass conservative interior penalty discontinuous Galerkin methods, of arbitrary order. We remark that the proof of existence and uniqueness of the discrete solution, which requires a small data condition, is not straightforward and follows a technical and convoluted argument, which can be traced back to [10]. The proof for the error estimates is also non-standard in the sense that one has to carefully bound terms with the appropriate norms. To our knowledge, the current paper is the first one to analyze convergence of a scheme for the time-dependent coupled case with the Beavers–Joseph–Saffman equations. While the obtained theoretical results are what we would expect, one important contribution of this paper is on the analysis itself. Our analysis is valid in 2D and 3D, and allows for rough (non-smooth) interface.

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An outline of the paper follows. Section 2 introduces the equations and the discrete spaces. The following section describes the numerical method, and is followed by a section on useful inequalities. Section 5 contains the proof for existence and uniqueness of the numerical solution. A priori error estimates are derived next. They are confirmed by numerical tests in Section 7. Concluding remarks follow.

2. Model problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded connected domain with a Lipschitz-continuous boundary, partitioned into two disjoint subdomains, Ω_1 and Ω_2, so that $\Omega = \Omega_1 \cup \Omega_2$. The free flow region and the porous medium are Ω_1 and Ω_2 respectively. We assume that each subdomain Ω_i also has a Lipschitz-continuous boundary. Let Γ_{12} denote the interface between Ω_1 and Ω_2. The interface Γ_{12} may be a polygonal curve or surface and does not have to be smooth. Let Γ_i be the exterior boundary of Ω_i, $i = 1, 2$. The boundary Γ_2 is partitioned into two disjoint open sets, $\Gamma_2 = \Gamma_{2D} \cup \Gamma_{2N}$, and we assume that $|\Gamma_1| > 0$ and $|\Gamma_{2D}| > 0$, where $|\cdot|$ denotes the measure.

We denote by $\mathbf{n}_{\Omega_i}$ the exterior unit vector normal to Γ_i and by $\mathbf{n}_{12}$ the unit normal vector to Γ_{12} pointing from Ω_1 to Ω_2. In the case $d = 3$, we choose a pair of orthonormal tangent vectors, $\boldsymbol{\tau}_{12}^j$, $j = 1, 2$, on the tangent plane to Γ_{12}. For two-dimensional domains, the vector $\boldsymbol{\tau}_{12}^1$ is the unit tangent vector to Γ_{12}.

We are interested in studying a fully discrete discontinuous Galerkin approximation to the following equations posed in each subdomain (the equalities below hold almost everywhere in the domains or on the boundaries, according to the context):

$$\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (2\mu \mathbf{D}(\mathbf{u}) - p_1 \mathbf{I}) + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{f}_1, \quad \text{in } \Omega_1 \times (0, T), \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega_1 \times (0, T), \tag{2}$$

$$-\nabla \cdot \mathbf{K} \nabla p_2 = f_2, \quad \text{in } \Omega_2 \times (0, T), \tag{3}$$

with the symmetric gradient $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$. Eqs. (1), (2) represent the incompressible Navier–Stokes equations, where $\mathbf{u}$ and p_1 are the fluid velocity and pressure respectively in Ω_1. The fluid viscosity is denoted by $\mu > 0$. Eq. (3) represents the single phase flow equation in a porous medium, where p_2 is the fluid pressure in Ω_2. The matrix $\mathbf{K}$ is the ratio of the permeability matrix to the fluid viscosity. It is assumed to be symmetric positive definite, with eigenvalues uniformly bounded above and bounded below away from zero. This system is complemented by the boundary and the interface conditions below. First we prescribe standard conditions on Γ_i:

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \Gamma_1 \times (0, T), \tag{4}$$

$$p_2 = 0, \quad \text{on } \Gamma_{2D} \times (0, T), \tag{5}$$

$$\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{\Omega_2} = 0, \quad \text{on } \Gamma_{2N} \times (0, T). \tag{6}$$

On the interface Γ_{12}, we prescribe the following interface conditions:

$$\mathbf{u} \cdot \mathbf{n}_{12} = -\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}, \quad \text{on } \Gamma_{12} \times (0, T), \tag{7}$$

$$\big((-2\mu \mathbf{D}(\mathbf{u}) + p_1 \mathbf{I}) \mathbf{n}_{12}\big) \cdot \mathbf{n}_{12} = p_2, \quad \text{on } \Gamma_{12} \times (0, T), \tag{8}$$

$$\mathbf{u} \cdot \boldsymbol{\tau}_{12}^j = -2\mu G^j \big(\mathbf{D}(\mathbf{u}) \mathbf{n}_{12}\big) \cdot \boldsymbol{\tau}_{12}^j, \ 1 \le j \le d-1, \quad \text{on } \Gamma_{12} \times (0, T), \tag{9}$$

where

$$G^j = \frac{\mu \alpha}{(\mathbf{K} \boldsymbol{\tau}_{12}^j, \boldsymbol{\tau}_{12}^j)^{1/2}}.$$

The interface conditions (7)–(9) have been discussed extensively in the literature for the steady-state coupling of porous media and free flows [11,12,7,13,2]. The condition (9) is the Beavers–Joseph–Saffman condition. We note that $\alpha > 0$ is a given constant, usually obtained from experimental data.

Finally, to simplify the discussion, we prescribe a zero initial condition:

$$\mathbf{u} = \mathbf{0}, \quad \text{in } \Omega_1 \times \{0\}. \tag{10}$$

The relevant spaces for the exact solution $(\mathbf{u}, p_1, p_2)$ are

$$\mathbf{X} = \{\mathbf{v} \in H^1(\Omega_1)^d\, ;\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}, \tag{11}$$

$$M_1 = L^2(\Omega_1), \tag{12}$$

$$M_2 = \{q \in H^1(\Omega_2)\, ;\ q = 0 \text{ on } \Gamma_{2D}\}. \tag{13}$$

For the discretization, we assume that the domain has a polygonal or polyhedral boundary, according to the dimension, so that it can be entirely triangulated. This simplifies substantially the numerical analysis. Let $\mathcal{E}_i^h$ be a regular family of