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Rapid evaluation of two-dimensional retarded time integrals

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Abstract. We present two methods for rapid evaluation of two-dimensional retarded time integrals. For example, such integrals arise as the $z = 0$ trace $U(t, x, y, 0)$ of a solution $U(t, x, y, z)$ to $3 + 1$ wave equation $\Box U = -2f(t, x, y)\delta(z)$ forced by a "sheet source" at $z = 0$. The spatial Fourier transform of a two-dimensional retarded time integral involves a temporal convolution with the zeroth order Bessel function $J_0(t)$. Appealing to work by Alpert, Greengard, and Hagstrom and by Xu and Jiang on rational approximation in the Laplace-transform domain, our first method relies on approximation of $J_0(t)$ as a sum of exponential functions. We achieve approximations with double precision accuracy near $t \simeq 0$, and maintain single precision accuracy out to $T \simeq 10^8$. Our second method involves evolution of the 3+1 wave equation in a "thin block" above the sheet, adopting the radiation boundary conditions of Hagstrom, Warburton, and Givoli based on complete plane wave expansions. We review their technique, present its implementation for our problem, and present new results on the nonlocal spacetime form of radiation boundary conditions. Our methods are relevant for the sheet-bunch formulation of the Vlasov-Maxwell system, although here we only test methods on a model problem, a Gaussian source following an elliptical orbit. Our concluding section discusses the complexity of both methods in comparison with naive evaluation of a retarded-time integral.

Keywords: retarded time integral, rational approximation, radiation boundary conditions, initial boundary value problem, Vlasov-Maxwell system, accelerator beam physics.

2010 MSC: 35L05, 41A20, 65A05, 65M99, 65Z05, 78M25

1. Introduction and preliminaries

1.1. **Introduction.** This article presents two strategies for fast numerical evaluation of the retarded-time integral

$$F(t, \mathbf{x}) := \frac{1}{2\pi} \int_{|\mathbf{x}-\mathbf{x}'| \le t} d\mathbf{x}' \frac{f(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \tag{1}$$

where $\mathbf{x} = (x, y)$, $d\mathbf{x}' = dx'dy'$, and we have included the $(2\pi)^{-1}$ in (1) for later convenience. Both strategies are useful when the integral must be evaluated over a range of spatial $\mathbf{x}$ and temporal t points. Of particular interest is the support of $F(t, \cdot)$ in the case when the support of $f(t, \cdot)$ is small for all past times. One scenario involving integrals of this form is the Maxwell equations with charge and current densities that are confined to a "sheet" $z = 0$ and have small supports. For this scenario, the $z = 0$ restrictions of the electric field E_x, E_y and the magnetic field B_z components obey (1), in each case with $f(t, \mathbf{x})$ appropriately

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