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Reproducing kernel method for solving singularly perturbed differential-difference equations with boundary layer behavior in Hilbert space

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ABSTRACT

We consider singularly perturbed differential-difference equation, which contains negative shift in the differentiated term with boundary layer behavior. RKHSM (Reproducing Kernel Hilbert Space Method) without Gram–Schmidt orthogonalization process, is considered in the present paper. We decompose the domain of the problem into two subintervals. One of them has not the boundary layer and the other one has. The side of the interval in which the boundary layer exists is important. If the boundary layer of this problem exists on the left side of interval, the RKHSM will provide a proper approximation of solution, otherwise for the implement of RKHSM, we need to change the variable of the singularly perturbed problem to shift the boundary layer region.

1. Introduction

Singularly perturbed problems that have boundary layer behavior are equations in which the highest derivative is multiplied by a perturbation parameter. In general, obtaining a proper approximation of the solution to the singularly perturbed problem that has the boundary layer, is difficult. When the perturbation parameter (ε) is small, the boundary layers of the considered problem are defined at the small regions from the problem domain, due to the fact that obtaining an approximate solution to the problem is difficult. For this reason, in order to obtain a proper approximation of the solution to the problem, we must consider the problem in three subintervals, and solve it using the RKHSM in each of the subintervals. Differential-difference equations arise in many branches of sciences for various practical problems in biomechanics and physics, economical, physiological, physio-chemical, biophysical, biology and control theory [1–5]. RKHSM is a powerful scheme to solve some mathematical problems, such as ODEs (Ordinary Differential Equations), PDEs (Partial Differential Equations), and BVPs (Boundary Value Problems). Many researchers use this method with Gram–Schmidt orthogonalization algorithm, but if we can implement this method without Gram–Schmidt algorithm, CPU time in this method will be reduced. For more details, see [6–8].

The accuracy of approximate solution in RKHSM depends on other factors. In this paper, we will discuss some of them. For example, the type of the points selection at the interval that the problem is defined on it and also the distance function. Obviously, the type of picking up points in each of the subintervals is different and it is known that on an equidistant mesh for singularly perturbed problems that have boundary layer behavior, RKHSM does not provide an accurate approximation,

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unless mesh points have an exponential property. In singularly perturbed problems after dividing the interval of our problem into two subintervals, in the subinterval that boundary layer exists, mesh points should have an exponential property and then RKHSM will provide a proper approximation of solution. In singularly perturbed problems with boundary layer behavior, finding the subinterval that boundary layer exists on it, is important to obtain a proper approximation of solution. Other factors such as the smoothness order of reproducing kernel space is important in the accuracy of approximate solution. For more details, see [9].

The following strategy is adopted in the rest of the paper. In Section 2, we introduce the singularly perturbed differential-difference equation. In Section 3, we provide all of the reproducing kernel spaces and their kernels that we need after dividing the problem domain and we provide the reproducing kernel Hilbert space method without Gram–Schmidt orthogonalization process and for this reproducing kernel method, we prove the convergence theorems and, we explain how to implement the reproducing kernel Hilbert space method at the divided intervals for the singularly perturbed differential-difference equation. In Section 4, we provide an error analysis for the proposed method. In Section 5, two numerical examples have been solved using the proposed method to show the validity of the results in this paper. Section 6 ends the paper with a brief conclusion.

2. Statement of the problem

We consider singularly perturbed differential-difference equations with boundary layer behavior in the neighborhood of $x = 0$ or $x = 1$ and a negative shift in the differentiated term as follows

$$\begin{cases} \varepsilon u''(x) + p(x)u'(x-\delta) + q(x)u(x) = f(x), \quad x \in [0, 1], \\ \qquad u(x) = \phi(x), \quad x \in [-\delta, 0], \; u(1) = \gamma, \end{cases} \tag{1}$$

where ε and δ are small parameters and $0 < \varepsilon \ll 1$, $0 < \delta \ll 1$. The functions $p(x)$, $q(x)$, $f(x)$, $\phi(x)$ are smooth functions and γ is a constant and $q(x) \le -\theta < 0$, $p(x) \ge M > 0$, where θ, M are generic positive constants. By using Taylor's series expansion of the term $u'(x-\delta)$ around x, we have

$$u'(x-\delta) \approx u'(x) - \delta u''(x),$$

see [10,11]. Thus, problem (1) is converted to

$$\begin{cases} (\varepsilon - \delta p(x))u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \quad x \in [0, 1], \\ \qquad u(0) = \phi(0), \qquad u(1) = \gamma. \end{cases} \tag{2}$$

If $p(x) > 0$ and $(\varepsilon - \delta p(x)) > 0$, $\forall x \in [0, 1]$ then the boundary layer of problem (2) will exist on the left side and if $p(x) < 0$ or $p(x) > 0$ and $(\varepsilon - \delta p(x)) < 0$, $\forall x \in [0, 1]$ then the boundary layer of problem (2) will exist on the right side of the interval $[0, 1]$.

3. Description of the method

Consider $d = |k(\varepsilon \pm \delta)Ln(kN)|$, where k is a positive real number and N is the number of points throughout the interval $[0, 1]$. In this section, and in the next section, we will consider two cases for problem (1).

Case 1. If the boundary layer of problem (1) exists on the left side, then we divide the interval $[0, 1]$ into two subintervals $[0, d]$ and $[d, 1]$. The subinterval $[d, 1]$ for which we have no boundary layer and subinterval $[0, d]$ for which we have boundary layer. The subinterval in which the boundary layer does not exist, is called the regular region and the subinterval in which the boundary layer exists, is called the boundary layer region. In this case, we construct reproducing kernel for each of these subintervals according to the boundary conditions of the problem (1) in space $W_2^3[0, d]$ and $W_2^3[d, 1]$. In addition, in this case that the boundary layer exists on the left side we do not need the variable change in the problem (1) for the subinterval $[0, d]$, because reproducing kernel Hilbert space provides a proper approximation of solution for the problem (1).

3.1. *Solution of the regular region problem for $x \in [d, 1]$*

Consider problem (2) in the regular region

$$\begin{cases} (\varepsilon - \delta p(x))u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \\ \qquad x \in [d, 1], \qquad u(1) = \gamma. \end{cases} \tag{3}$$

After the homogenization of the boundary condition $u(1) = \gamma$, in order to solve the problem (3) by using the RKHSM, first we construct reproducing kernel space $W_2^3[d, 1]$.