# Accepted Manuscript


An efficient quadrature rule on the Cubed Sphere

Brice Portelenelle, Jean-Pierre Croisille

PII: S0377-0427(17)30327-8
DOI: http://dx.doi.org/10.1016/j.cam.2017.06.027
Reference: CAM 11205

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 2 October 2016
Revised date : 10 May 2017

Please cite this article as: B. Portelenelle, J.-P. Croisille, An efficient quadrature rule on the Cubed Sphere, *Journal of Computational and Applied Mathematics* (2017), http://dx.doi.org/10.1016/j.cam.2017.06.027

# AN EFFICIENT QUADRATURE RULE ON THE CUBED SPHERE

BRICE PORTELENELLE AND JEAN-PIERRE CROISILLE †‡

Abstract. A new quadrature rule for functions defined on the sphere is introduced. The nodes are defined as the points of the Cubed Sphere. The associated weights are defined in analogy to the trapezoidal rule on each panel of the Cubed Sphere. The formula enjoys a symmetry property ensuring that a proportion of 7/8 of all Spherical Harmonics is integrated exactly. Based on the remaining Spherical Harmonics, it is possible to define modified weights giving an enhanced quadrature rule. Numerical results show that the new quadrature is competitive with classical rules of the literature. This second quadrature rule is believed to be of interest for applied mathematicians, physicists and engineers dealing with data located at the points of the Cubed Sphere.

*Keywords: Quadrature rule on the sphere ; Cubed Sphere ; Spherical Harmonic.*
*Mathematics Subject Classification: 33C55, 41A55, 65D30, 65D32*

## 1. Introduction

In this paper we consider quadrature rules for functions defined on the sphere. Let $\mathbb{S}^2$ be the unit sphere, and let $\boldsymbol{x} \in \mathbb{S}^2 \mapsto f(\boldsymbol{x})$ be a regular function. A quadrature rule $Q(f)$ is defined by

$$I(f) = \int_{\mathbb{S}^2} f(\boldsymbol{x}) d\sigma(\boldsymbol{x}) \simeq \sum_{p=1}^{P} w_p f(\boldsymbol{x}_p) = Q(f), \tag{1}$$

where $\boldsymbol{x}_p \in \mathbb{S}^2$ are the nodes and $w_p$ are the weights. The quest for rules of the form (1) has been a longstanding topic of interest. The classical setup of the problem consists in finding a minimal number $\mathcal{P}$ of nodes $x_p$ with the associated weights $w_p$ for (1) to be as exact as possible. More precisely, the problem is to determine the location on the sphere of a minimal number of nodes for the largest number of Spherical Harmonics to be exactly integrated. A classical reference is [13]. Recent works on this topic include [16, 11, 1, 8]. For a general presentation of the problem we refer to the review chapters [9, Chap. 40] and [2, Chap. 5].

Here we consider the problem with a slightly different point of view. Over the past 20 years, the Cubed Sphere (see Fig.1) has become a popular spherical grid among researchers dealing with mathematical or physical models. In particular, in numerical climatology, the Cubed Sphere serves for various numerical schemes for time-dependent climate models on the sphere [12, 3, 17]. In this context, accurately evaluating averaged quantities over the sphere such as mass, momentum, energy or total vorticity is particularly important. This is in particular the case for the finite difference scheme introduced in [5, 6]. This scheme uses discrete unknowns located at the points of the Cubed Sphere. To use with this scheme, it is important to have a quadrature rule with nodes $\boldsymbol{x}_p$ selected as the points of the Cubed Sphere.

To fully determine such a rule, it remains to identify a set of suitable weights $w_p$. A basic observation is that a particularly simple set of weights $w_p$, described hereafter, provides a rule (1), which is exact for a proportion of 7/8 of *all* Spherical Harmonics. Furthermore, for the remaining

*Date*: July 1, 2017.
This work was performed during the master thesis of the first author at the Univ. Lorraine, Math. Dept., Institut Elie Cartan de Lorraine in Metz, France, during the second semester of the academic year 2015-2016.