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Barrier option pricing under the 2-hypergeometric stochastic volatility model

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HIGHLIGHTS

- We investigate the pricing of barrier options under stochastic volatility.
- Using an asymptotic expansion method, an explicit pricing formula is derived.
- The convergence of the asymptotic solution is proved.
- The evaluation of the asymptotic prices is fast and suitable for practical uses.
- Our numerical examples demonstrate that the approach leads to a small error.

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ABSTRACT

We investigate the pricing of financial options under the 2-hypergeometric stochastic volatility model. This is an analytically tractable model that reproduces the volatility smile and skew effects observed in empirical market data.

Using a regular perturbation method from asymptotic analysis of partial differential equations, we derive an explicit and easily computable approximate formula for the pricing of barrier options under the 2-hypergeometric stochastic volatility model. The asymptotic convergence of the method is proved under appropriate regularity conditions, and a multi-stage method for improving the quality of the approximation is discussed. Numerical examples are also provided.

1. Introduction

Barrier options are options whose payoff does not depend only on the value of the underlying asset at maturity, but also on whether the path of the asset's price touches a given barrier level during the lifetime of the option. These options, which constitute one of the oldest types of exotic options, have become increasingly popular in the financial derivative industry because they allow for more flexible payoff schemes than plain vanilla options. It is thus important to construct good barrier option pricing models which are able to reproduce the features observed in real market data.

The simplest model for the pricing of financial derivatives is the Black and Scholes model, in which the price of all the standard barrier call and put options can be written in closed form. However, it is widely known that the strong assumptions

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of this model are unrealistic. In particular, the constant volatility assumption is clearly incompatible with the so-called smile and skew patterns which are generally present in empirical option prices.

A natural way to address this issue is to introduce randomness in the volatility. For this reason, option pricing under stochastic volatility has been the subject of a great deal of research in recent years. Here we focus on the 2-hypergeometric stochastic volatility model, which was introduced by Da Fonseca and Martini [1] as a model which ensures that the volatility is strictly positive — this is an important property which is not present in some other well-established stochastic volatility models. In a very recent paper, Privault and She [2] demonstrated that, under this model, a closed-form asymptotic vanilla option pricing formula can be determined through a regular perturbation method. This is a notable result because their formulas are analytically very simple, which is rarely the case in models with stochastic volatility: as discussed by Zhu [3], the higher complexity of these models usually yields the need for rather sophisticated numerical implementations.

The literature on barrier option pricing methods is extensive. Exact closed-form pricing formulas have been derived for only a few models other than that of Black and Scholes, none of which reproduces satisfactorily the market phenomena. (For explicit formulas under one-dimensional models see e.g. Davydov and Linestky [4], Hui and Lo [5]; for an explicit solution under the Heston stochastic volatility model see Lipton [6].) Given the unavailability of explicit formulas, to price barrier options under more complex models one needs to resort to numerical methods. The main approaches are the use of numerical partial differential equation (PDE) techniques and of Monte Carlo methods (we refer the reader to the books of Seydel [7] and of Glasserman [8]), which are often combined with other analytical or numerical techniques (for recent work see for instance Zhang et al. [9], Guardasoni and Sanfelici [10]). Unfortunately, the computational times are nowadays still largely incompatible with the demands of the financial industry.

An alternative strategy for pricing under more general models is to derive approximate (or asymptotic) analytical solutions: this has been proposed not only for vanilla options (cf. Privault and She [2]) but also for barrier options, see e.g. Fouque et al. [11], Alòs et al. [12]. These asymptotic methods are intrinsically computationally much less expensive than the numerical methods mentioned in the previous paragraph. Indeed, numerical PDE techniques usually rely on space–time discretization and on the numerical solution of linear systems of high dimension, while Monte Carlo methods require the simulation of a large number of sample paths on a suitably fine time grid; on the other hand, the asymptotic techniques only require the computation of a few integrals (the number of such integrals is small and does not depend on the discretization). Thus the key question when dealing with asymptotic solutions is whether they are sufficiently exact for practical purposes.

Despite the vast body of work in this area, the pricing of barrier and other exotic options under the 2-hypergeometric model has to our knowledge never been studied. Motivated by this, we extend the regular perturbation approach of Privault and She in order to derive an asymptotic pricing formula for barrier-type options. We show that, for a given class of nonconstant barrier functions, an explicit asymptotic formula can be obtained and its convergence can be proved with the help of the Feynman–Kac theorem for Cauchy–Dirichlet problems for parabolic PDEs. Given that in general our class of barrier functions does not include constant functions, the choice of a nonconstant barrier function which approximates a certain constant barrier level is discussed. We also present some numerical examples which indicate that the accuracy of our asymptotic formulas is high enough for the applications.

This paper is organized as follows. In Section 2 we introduce the class of barrier options which we consider, and we present the formulation of the barrier option pricing problem under a generic stochastic volatility model. Section 3, the main section of this paper, develops the asymptotic pricing approach for barrier-type options: the first-order small volatility expansion is carried out in Section 3.1, the explicit expressions for the zero and first-order terms are derived in Sections 3.2 and 3.3 respectively, the proof of the convergence of the asymptotic solution is provided in Section 3.4, and a generalization of the method to a wider class of barrier functions is given in Section 3.5. In Section 4 we present some numerical results to corroborate our theoretical findings. Section 5 summarizes the main conclusions. The appendices contain some auxiliary results.

2. Barrier option pricing under stochastic volatility

This work focuses on the pricing of *down-and-out call* (DOC) options, which are one of the eight types of standard barrier options. The techniques used in this paper may also be applied to other types of barrier options, such as options with up barriers or put payoffs. The payoff of a DOC option with maturity T is

$$(S_T - K)^+ \, \mathbb{1}_{\{S_t > H \text{ for all } 0 \le t \le T\}},$$

i.e, it has the usual vanilla call payoff if the asset price process S does not go below the barrier H during the lifetime of the option, and it is worthless otherwise. The DOC option is said to be *regular* if $K \ge H$ and *reverse* if $K < H$. If a barrier function $H(t)$ is considered instead of a constant barrier H, the DOC option is called *time-dependent*.

For the sake of generality, let us begin by assuming that the asset process is governed (under the physical measure $\mathbb{P}$) by a Markovian stochastic volatility model of the form

$$\begin{aligned} dS_t &= \mu(t, S_t) S_t \, dt + g(V_t) S_t \, dW_t^1 \\ dV_t &= a(t, V_t) \, dt + b(t, V_t) \, dW_t^2 \end{aligned} \tag{1}$$

where S is the asset price process, V is the volatility process, W^1 and W^2 are Brownian motions with correlation $\rho \neq \pm 1$, and g is a smooth, positive and increasing function. This is a general family of models which includes the 2-hypergeometric model addressed in the main section of this paper, as well as the Heston model and other popular stochastic volatility models.