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FINITE ELEMENT METHOD FOR A SPACE-FRACTIONAL ANTI-DIFFUSIVE EQUATION

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ABSTRACT. The numerical solution of a nonlinear and space-fractional anti-diffusive equation used to model dune morphodynamics is considered. Spatial discretization is effected using a finite element method whereas the Crank-Nicolson scheme is used for temporal discretization. The fully discrete scheme is analyzed to determine stability condition and also to obtain error estimates for the approximate solution. Numerical examples are presented to illustrate convergence results.

1. INTRODUCTION

We consider the Fowler equation [7]

$$\partial_t u(t,x) + \partial_x \left(\frac{u^2}{2}\right)(t,x) - \partial_x^2 u(t,x) + \mathcal{I}[u](t,x) = 0, \quad x \in \mathbf{R}, t > 0, \tag{1.1}$$

where $\mathcal{I}$ is a nonlocal operator defined as follows: for any Schwartz function $\varphi \in \mathcal{S}(\mathbf{R})$ and any $x \in \mathbf{R}$,

$$\mathcal{I}[\varphi](x) := \int_0^{+\infty} |\xi|^{-\frac{1}{3}} \varphi''(x-\xi)\, d\xi. \tag{1.2}$$

The Fowler equation was introduced to model the formation and dynamics of sand structures such as dunes and ripples [7]. This equation is valid for a river flow over an erodible bottom $u(t,x)$ with slow variation. Its originality resides in the nonlocal term, which is anti-dissipative, and can be seen as a fractional Laplacian of order $4/3$. Indeed, it has been proved in [2] that

$$\mathcal{F}(\mathcal{I}[\varphi])(\xi) = -4\pi^2 \Gamma(\frac{2}{3}) \left(\frac{1}{2} - i\mathrm{sgn}(\xi)\frac{\sqrt{3}}{2}\right) |\xi|^{4/3} \mathcal{F}(\varphi)(\xi),$$

where Γ is the gamma function and $\mathcal{F}$ denotes the Fourier transform.
Therefore, this term has a deregularizing effect on the initial data but the instabilities produced by the nonlocal term are controled by the diffusion operator $-\partial_x^2$ which ensures the existence and the uniqueness of a smooth solution [2]. We then always assume that there exists a sufficiently regular solution $u(t,x)$.
The use of Fourier transform is a natural way to study this equation but it also can be useful to consider the following formula:
for all $r > 0$ and all $\varphi \in \mathcal{S}(\mathbf{R})$,

$$\mathcal{I}[\varphi](x) \quad = \quad \mathcal{I}_1[\varphi](x) + \mathcal{I}_2[\varphi](x), \tag{1.3}$$

Key words and phrases. Fractional anti-diffusive operator, finite element method, Crank-Nicolson scheme, stability, error analysis.