



Convergence analysis for GMsFEM approximation of elliptic eigenvalue problems



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ABSTRACT

In this paper, we analyze the approximation of elliptic eigenvalue problems using generalized multiscale finite element method (GMsFEM) and get error estimates for eigenfunctions and eigenvalues. For the case of simple eigenvalues, the approximation errors for eigenfunctions are considered in both energy error and L^2 error. The derived error estimates clearly give the relation between the errors and the coarse mesh size, local multiscale enrichment and the corresponding eigenvalues. The convergence analysis shows that the approximation of eigenvalue problems using GMsFEM does not depend on the contrastness of the coefficient when the diffusion coefficient is highly heterogeneous. A few numerical examples are presented to illustrate the performance of the GMsFEM approximation and the theoretic analysis for the eigenvalue problems.

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1. Introduction

In many scientific and engineering applications, it is very important to compute the eigenfunctions and eigenvalues for differential equations. In this paper, we solve the elliptic eigenvalue problems with rough coefficients using Generalized Multiscale Finite Element Method (GMsFEM). The general theory of the numerical eigenvalue problems can be found in the literatures [1–4].

For the eigenvalue problem of partial differential equations, a popular numerical method is using finite element method. The work [5] presents the finite element discretization of eigenvalue problems for an elliptic partial differential operator and provides its error analysis. In [1], there is a direct proof of convergence for eigenvalues and eigenfunctions of the Laplace equation approximated with piecewise linear elements, which is based on basic properties of the Rayleigh quotient. In many works, the priori error estimates are based on the hypothesis that the mesh-size is sufficiently small. Knyazev and Osborn in [6] have overcome this difficulty and presented the first truly a priori error estimate for symmetric eigenvalue problems. For the a posteriori and a priori error analysis for finite element approximation of self-adjoint elliptic eigenvalue problems, we can refer to [7–10]. They have given the error analysis of multiple eigenvalues and simple eigenvalues. The works have also proved that a posteriori error estimates for a general nonsymmetric eigenvalue problem are obtained without the requirement of H^2 -regularity. There are many methods to compute eigenvalues and eigenfunctions for a matrix, such as Jacobi method [11], the power method [12], Arnoldi-type iterations and QR-algorithm [13–15]. Although those methods are efficient for solving the matrix eigenvalue problems, there exists some limitation when the matrix is derived through the discretization of multiscale models. In these situations, the associated matrix may be very large-scale if standard finite

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element is used to resolve all scales of the models. In these circumstances, it may be infeasible to compute the eigenvalue problems directly.

In order to avoid the application of any eigenvalue solver to the very fine scale discretization (2.8) directly, multigrid methods can be helpful. As early as 1985, W. Hackbusch in [16] has proposed a multigrid iteration, which approximates simultaneously eigenvalues and the associated vectors. The matrix may be unsymmetric or even not diagonalizable. The algorithm is based on a Newton iteration converging to a sub-matrix of the Schur normal form. Based on the finite element method and iteration, W. Hackbusch [17], Xie [18], Xu et al. [19] have proposed multigrid methods to solve eigenvalue problems. These methods showed that the solution of the eigenvalue problem on a fine grid is reduced to the solution on a much coarser grid, which only needs a small amount of computational work and the asymptotically optimal accuracy is well maintained. There are also other methods to solve the eigenvalue problem without iteration.

During the last decades, various multiscale methods have been developed [20–24]. The main idea of multiscale method is to divide the fine scale problem into many local problems and use the solutions of the local problems to form a coarse scale equation (refer to [24]). Among these multiscale methods, multiscale finite element method (MsFEM) [21] is one of the efficient approaches. The basic idea of MsFEM is to incorporate the small-scale information to multiscale basis functions and capture the impact of small-scale features on the coarse-scale through a variational formulation. MsFEM can be used to efficiently compute multiscale eigenvalue problems and avoid computing the large-scale problem directly. The authors in [25] discussed the multiscale finite element computation of a Steklov eigenvalue problem with rapidly oscillating coefficients and derived corresponding convergence results. Axel et al. [26] considered benchmark multiscale eigenvalue problems by computing a low-dimensional generalized (possibly mesh free) finite element space that preserves the lowermost eigenvalues in a superconvergent way. After solving the corresponding low-dimensional algebraic eigenvalue problem, the approximate eigenpairs are then obtained.

Although these multiscale methods can give accurate solutions in some cases, there are applications where these approaches are inadequate and some enhancements are needed. In particular, one may need to use multiple basis functions per coarse block to capture complex heterogeneities and continuum scales in the solutions. To this end, a generalized multiscale method has been developed in the framework of generalized finite element method [27,28] by the name of generalized multiscale finite element method (GMsFEM) [29,30]. The construction of multiscale basis functions relies on the choice of suitable snapshot space and dimension reduction procedure by solving local spectral problems. In this paper, we will use GMSFEM to solve elliptic eigenvalue problems with rough coefficients. The computation for the eigenvalue problems is performed on a coarse grid. This leads to an algebraic eigenvalue problem in coarse scale. It is desirable for any usual linear eigenvalue solvers. Using the multiscale basis functions, we can downscale the coarse scale eigenfunctions to fine scale eigenfunctions, which provide accurate approximation of eigenfunctions. We present rigorous convergence analysis for GMSFEM approximation of elliptic eigenvalue problems. The theoretic analysis and numerical results show that the convergence is independent of the contrastness of the rough coefficient. As far as we know, there are very few literatures to rigorously study the eigenvalue problems using GMSFEM.

The paper is organized as follows. In Section 2, we give some preliminaries and a short introduction to generalized multiscale finite element method. Section 3 is devoted to error estimates for eigenfunction and eigenvalue using GMSFEM approximation. In this section, we give the error analysis of eigenfunction in the energy norm and L^2 -norm. By the definition of Rayleigh quotient, convergence analysis is presented for the eigenvalues as well. In Section 4, some numerical examples are given to confirm the theoretical analysis. Finally, some conclusions and comments are outlined in the last section.

2. Preliminaries and GMSFEM

In this section, we first introduce some notations used for the paper and then present GMSFEM. Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be a polyhedral, bounded domain with Lipschitz boundary $\partial\Omega$. In the paper, we consider the following elliptic eigenvalue problem,

$$\begin{cases} -\operatorname{div}(\kappa(x)\nabla u) = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Here the coefficient $\kappa(x) \in L^\infty(\Omega)$ is uniformly bounded below and above, i.e., there exists $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq \kappa(x) \leq \beta, \quad \forall x \in \Omega. \quad (2.2)$$

The ratio $\frac{\beta}{\alpha}$ characterizes the contrastness of the diffusion coefficient.

Let $W_p^k(\Omega)$ be the usual Sobolev space with norm $\|\cdot\|_{W_p^k(\Omega)}$. In particular, we write $H^k(\Omega) = W_2^k(\Omega)$ and $L^p(\Omega) = W_p^0(\Omega)$. We denote

$$H^1(\Omega) := \{v \in L^2(\Omega) : \nabla v \in L^2(\Omega)\}$$

and

$$H_0^1(\Omega) := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

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