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Global–local model reduction for heterogeneous Forchheimer flow

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ABSTRACT

In this paper, we propose a mixed Generalized Multiscale Finite Element Method (GMsFEM) for solving nonlinear Forchheimer flow in highly heterogeneous porous media. We consider the two term law form of the Forchheimer equation in the case of slightly-compressible single-phase flows. We write the resulting system in terms of a degenerate nonlinear flow equation for pressure when the nonlinearity depends on the pressure gradient. The proposed approach constructs multiscale basis functions for the velocity field following Mixed-GMsFEM as developed in Chung et al. (2015). To reduce the computational cost resulting from solving nonlinear system, we combine the GMsFEM with Discrete Empirical Interpolation Method (DEIM) to compute the nonlinear coefficients in some selected degrees of freedom at each coarse domain. In addition, a global reduction method such as Proper Orthogonal Decomposition (POD) is used to construct the online space to be used for different inputs or initial conditions. We present numerical and theoretical results to show that in addition to speeding up the simulation we can achieve good accuracy with a few basis functions per coarse edge. Moreover, we present an adaptive method for basis enrichment of the offline space based on an error indicator depending on the local residual norm. We use this enrichment method for the multiscale basis functions at some fixed time levels. Our numerical experiments show that these additional multiscale basis functions will reduce the current error if we start with a sufficient number of initial offline basis functions.

1. Introduction

Solving nonlinear flows in porous media, such as the Forchheimer flow, is very important to many recent engineering and applied sciences research. However, solving nonlinear flows with high contrast geological parameters, needs large computational cost. Some coarsening and up-scaling methods have been proposed for the linear Darcy case (e.g., [1–6]). The extensions of these methods to nonlinear flows, such as Forchheimer flow, have been used to reduce the computational complexity. In [7], a spacial nonlinear upscaled Forchheimer form is used to simplify the calculation. In [8], a local–global up-scaling technique is iteratively used. A new formulation for Forchheimer flow is used in [9]. The authors reduced the original system of equations for pressure and velocity to one nonlinear equation for pressure only. This equivalent form is obtained by using monotone nonlinear permeability function of the gradient of pressure.

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On the other hand, the generalized multiscale finite element method (GMsFEM) has been recently used in some active research purposed to efficiently and cheaply solve linear/nonlinear system of PDEs (see [10–18]). In GMsFEM one constructs multiscale basis functions and solves the problem on a coarse grid. Typically, GMsFEM is referred to as a local reduction technique and is divided into two main stages:

- *Offline stage*
  In this stage we construct:
  - Snapshot space that contains the complete set of solutions of local problems for arbitrary parameters with all possible boundary conditions up to fine-grid resolution.
  - Offline space that is constructed via a proper spectral decomposition of the snapshot space.
- *Online stage*
  A reduced dimensional space is constructed to be used for any input parameter to solve the problem in the coarse grid. In [14,12], the Proper Orthogonal Decomposition (POD) modes (see [19–28]) are used to construct the online basis functions.

However, computing the GMsFEM solution in the case of nonlinear system involves fine grid calculations for the nonlinear coefficients. In such case (as in [14,11]) we combine GMsFEM with Discrete Empirical Interpolation Method (DEIM) to perform fine grid computations.

In [13], the mixed GMsFEM was presented for solving mixed framework of flow in heterogeneous media where the conservation of mass is essential. The authors considered the *linear* case and constructed the velocity field for the snapshot and offline spaces and used piecewise constant basis functions to approximate the pressure. In [15], two adaptive algorithms were developed to enrich the offline space resulting from applying the mixed GMsFEM presented in [13]. The first adaptive method is called the offline adaptive method. This method is used to increase the number of basis functions locally in coarse regions based on an a-posteriori error indicator which depends on the local residual norm. The second adaptive method is the online adaptive method (see also [16,17,29,30]). The online adaptive method iteratively enriches the function space by adding new basis functions computed based on the residual of the previous solution. The analysis of these studies show that by using sufficient number of initial offline basis functions, one can guarantee the additional online multiscale basis functions will reduce the error independent of the contrast of the medium.

In this paper, we consider *mixed* framework of the *nonlinear* Forchheimer flow. We follow our "Global–Local" method presented in [14] to obtain the multiscale solution. Precisely, to solve the nonlinear system of Forchheimer flow we apply the following:

1. Use the mixed GMsFEM to construct the offline (multiscale) space for velocity field and approximate the pressure in the space of piecewise constant functions.
2. Use DEIM algorithm to perform the fine grid computation of the nonlinear coefficients. We use local DEIM to approximate the nonlinear function locally at some degrees of freedom in each coarse grid in the offline problem. In the online problem we use global DEIM to compute the nonlinear function at some selected degrees of freedom in the global domain to speed up the simulation.
3. Use POD modes to construct the online space.
4. In order to improve the accuracy, we also use the adaptive online method at some fixed time levels to enrich the offline space and reduce the approximation error.

We present numerical examples to show that a significant reduction in our model is achieved with a few number of basis functions. The reduction, here, is obtained for the dimension of the solution space. For example, we reduced the dimension of the solution space from 540 to 4 in Example (6.1.1). This reduction yields to another reduction in the computational time. Therefore, the reduction in our method is for the dimension of the solution space and the computational time. Both numerical examples and theoretical results show that the accuracy and the convergence rate of the solution of our reduced system depend on:

1. The eigenvalue of the spectral problem used in the offline stage.
2. The number of offline basis functions.
3. The number of local and global DEIM used offline and online, respectively.
4. The number of POD modes (online basis functions).

The paper is organized as follows. In Section 2 we state the basic model problem. In Section 3 we introduce the global–local reduction methods, the mixed GMsFEM and the construction of the offline and the online spaces. The convergence analysis is given in Section 4. In Section 5 we present the online adaptive method. Numerical tests are presented in Section 6. We end the paper with a conclusion in Section 7.

2. Problem statement

In this work, we apply our recently developed method in [14] to a nonlinear parabolic PDE in mixed form. In particular, our interest is in solving the following form of Forchheimer equation

$$v + \beta(x)|v|v = -\frac{1}{\mu}\kappa(x)\nabla p, \tag{2.1}$$