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The polynomial Trefftz method for solving backward and inverse source wave problems

Chein-Shan Liu *

Center for Numerical Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing, Jiangsu 210098, China

Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung 202-24, Taiwan

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ABSTRACT

The Trefftz method is a truly meshless boundary-type method, because the trial solutions automatically satisfy the governing equation. In order to stably solve the high-dimensional backward wave problems and the one-dimensional inverse source problems, we develop a multiple-scale polynomial Trefftz method (MSPTM), of which the scales are determined a priori by the collocation points. The MSPTM can retrieve the missing initial data and unknown time varying wave source. The present method can also be extended to solve the higher-dimensional wave equations long-term through the introduction of a director in the two-dimensional polynomial Trefftz bases. Several numerical examples reveal that the MSPTM is efficient and stable for solving severely ill-posed inverse problems of wave equations under large noises.

1. Introduction

The wave motions are popular in many engineering problems, for instance the stress wave in solids, the wave propagation in fluids, the scattering problems of electromagnetic waves, and the sound wave propagation in media. There are many available methods for solving the wave equations [1–4]. Young et al. [5] and Gu et al. [6] have used a singularity-free Euler–Lagrangian method of fundamental solutions to solve this sort problems. Lin et al. [7] have developed a fast solver of the three-dimensional wave equation by using the sparse scheme of the method of fundamental solutions. Liu and Kuo [8] developed a multiple-direction Trefftz method for solving the three-dimensional wave equation in arbitrary spatial domain.

In this paper we first treat the one-dimensional wave equation and then we will extend the method to the higher-dimensional wave equations. Let us consider a one-dimensional wave equation:

$$u_{tt} = c^2 u_{xx}, \quad (x, t) \in \Omega := \{0 < x < \ell, \ 0 < t \le t_f\}, \tag{1}$$

$$u(0, t) = u_0(t), \quad u(\ell, t) = u_\ell(t), \quad 0 \le t \le t_f, \tag{2}$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \le x \le \ell, \tag{3}$$

where the subscripts x and t denote the partial differentials with respect to x and t, respectively. For the direct problem we specify boundary conditions and initial conditions on $\Gamma = \{x = 0, \ 0 \le t \le t_f\} \cup \{x = \ell, \ 0 \le t \le t_f\} \cup \{0 \le x \le \ell, \ t = 0\}$ in Eqs. (2) and (3).

* Correspondence to: Center for Numerical Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing, Jiangsu 210098, China.

E-mail address: csliu58@163.com.

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In contrast, for the backward wave problem we specify boundary conditions and one or two final time conditions on $\Gamma = \{x = 0,\ 0 \le t \le t_f\} \cup \{x = \ell,\ 0 \le t \le t_f\} \cup \{0 \le x \le \ell,\ t = t_f\}$. We consider one of the backward wave problem (BWP), which replaces Eq. (3) by the following conditions:

$$u(x, 0) = f(x), \quad u(x, t_f) = h(x), \quad 0 \le x \le \ell. \tag{4}$$

From the aspect of wave control, we have formulated an inverse wave problem by solving the problem that under what initial condition of $u_t(x, 0)$ the wave will vibrate with the desired quantity $h(x)$ at a time t_f. Presently, the method used to solve the BWP was the least-squares/shooting type and taking advantage of the control formalism and methodology [9,10]. The BWP as pointed out by Ames and Straugham [11] has important applications in the geophysics and optimal control theory. Bourgin and Duffin [12] and Abdul-Latif and Diaz [13] have proved that the BWP has a unique solution only when $ct_f/\ell =$ irrational. But when $ct_f/\ell =$ rational, the uniqueness is not satisfied.

The BWP in Eqs. (1), (2) and (4) is known to be ill-posed, since the uniqueness of solution may be invalid. Lesnic [14] has solved the BWP by using the Adomian decomposition method, and found that for the direct problem the convergence of the Adomian decomposition method is faster than that for the backward problem. Liu [15] has proposed a second-kind Fredholm integral regularization method to solve the BWP. Chang and Liu [16] have employed the backward group preserving scheme to solve the multi-dimensional BWP.

Sometimes we may encounter the problem that the wave source function in the wave equation is unknown. To resolve such an inverse wave source problem by using the control results of distributed systems has been developed by Yamamoto [17,18], Bruckner and Yamamoto [19], and Yamamoto and Zhang [20]. They used some observability estimates and controllability results, and using the multiplier method and the Hilbert uniqueness method deduced the uniqueness of solution and the reconstruction process. For the wave equation this method successfully leads to the reconstruction of point sources in a one-dimensional domain by boundary observations. In higher dimensional domains the reciprocity gap functional technique leads to the reconstruction of smoother unknown sources using boundary observations [21,22]. Komornik and Yamamoto [23] have determined the positions of point sources in a one-dimensional wave equation, and later the estimation results are extended to multi-dimension in [24]. Most researchers are concerned with the determination of point sources. Liu et al. [25] have developed a differencing technique to generate a small scale linear system to recover spatial-dependent or temporal-dependent wave sources. We will develop a novel and powerful method for the recovery of a time varying wave source in a one-dimensional wave equation by using two extra measurements of strains on two boundaries.

The remaining portion of the paper is arranged as follows. In Section 2 we use a small parameter and the Taylor series to derive a polynomial basis of the Trefftz method for the wave equation. Then, we use the boundary type meshless method to derive a linear algebraic equations (LAEs) system to determine the expansion coefficients, and point out its highly ill-conditioned behavior. In order to overcome the ill-condition, the multiple-scale polynomial Trefftz method (MSPTM) is introduced in Section 3, of which the scales are determined by the collocation points according to the concept of equilibrated matrix. Numerical examples of a direct problem and backward wave problems are given in Section 4, and in Section 5 we demonstrate how to use the MSPTM to solve the time-dependent inverse source problem (ISP). In Section 6 we extend the one-dimensional MSPTM to the higher-dimensional wave equations by a simple idea of using the director in the variable transformation, and the numerical examples are given. Finally, the conclusions are given in Section 7.

2. The polynomial Trefftz method

For the one-dimensional heat conduction equation:

$$u_t(x, t) = \alpha u_{xx}(x, t), \tag{5}$$

the following

$$p_n(x, t) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{\alpha^k t^k x^{n-2k}}{k!(n-2k)!} \tag{6}$$

is a heat polynomial, because it satisfies Eq. (5) automatically. The heat polynomials were introduced by Rosenbloom and Widder [26], and Widder [27–29]. For a more comprehensive discussion of the heat polynomial analogies for higher order evolution equations one can refer [30].

Like the heat polynomials for the heat conduction equation (5), we can derive the wave polynomials for the wave equation (1), of which we can take

$$u(x, t) = \exp(\gamma x + \gamma ct). \tag{7}$$

By assuming γ to be a small parameter and by using the Taylor series we have

$$u(x, t) = \sum_{n=0}^{\infty} p_n^f(x, t)\frac{\gamma^n}{n!}, \tag{8}$$