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Steklov approximations of harmonic boundary value problems on planar regions

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ABSTRACT

Error estimates for approximations of solutions of Laplace's equation with Dirichlet, Robin or Neumann boundary value conditions are described. The solutions are represented by orthogonal series using the harmonic Steklov eigenfunctions. Error bounds for partial sums involving the lowest eigenfunctions are found. When the region is a rectangle, explicit formulae for the Steklov eigenfunctions and eigenvalues are known. These were used to find approximations for problems with known explicit solutions. Results about the accuracy of these solutions, as a function of the number of eigenfunctions used, are given.

1. Introduction

This paper treats the approximation of solutions of Laplace's equations using harmonic Steklov eigenfunctions. The problems are posed on bounded planar regions Ω and the functions should satisfy either Dirichlet, Robin or Neumann boundary conditions

$$u = g \quad \text{or} \quad D_\nu u + b\,u = g \quad \text{on } \partial\Omega. \tag{1.1}$$

Here ν is the outward unit normal and $b \geq 0$ is a constant.

Results about orthogonal bases of the class of all finite energy harmonic functions $\mathcal{H}(\Omega) \subset H^1(\Omega)$ consisting of harmonic Steklov eigenfunctions are summarized below in Section 3. These functions have the property that they generate a basis of $\mathcal{H}(\Omega)$ and their boundary traces provide orthogonal bases of $L^2(\partial\Omega, d\sigma)$ and $H^{1/2}(\partial\Omega)$. This *spectral theory of trace spaces* is described in Auchmuty [1]. Here some results obtained in the computational approximation of harmonic functions using Steklov eigenfunctions associated with the lowest Steklov eigenvalues will be described. General results and error estimates for approximations are described in Sections 4 and 6. Computational results for some problems with exact solutions are described in Sections 5 and 7. The explicit formulae for the Steklov eigenvalues and eigenfunctions on rectangles of aspect ratio $h > 0$ are used here. For similar problems on general regions, further errors are introduced when approximations of the Steklov eigenfunctions and eigenvalues are used.

Existence–uniqueness theorems for these problems may be found in most texts that treat elliptic boundary value problems. From Weyl's lemma, the solutions are C^∞ on the region Ω. Under various assumptions on g, Ω and $\partial\Omega$ the

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solutions will be in specific Banach or Hilbert spaces of functions on Ω or $\overline{\Omega}$. For an excellent review of classical results about these problems see chapter 2 by Benilan in [2]. A function $u \in L^1(\Omega)$ is said to be an *ultraweak* solution of Laplace's equation provided it obeys

$$\int_\Omega u \, \Delta \varphi \; dxdy = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{1.2}$$

Such an ultraweak solution is a *classical solution* of Laplace's equation provided it is equivalent to a continuous function on $\overline{\Omega}$. There are classical, and other ultraweak, harmonic functions that are not in the standard Sobolev space $H^1(\Omega)$ — even when Ω is a disk.

General results about Steklov approximations of harmonic functions are described in Sections 3, 4 and 6. An algorithm for constructing a basis of the subspace of harmonic functions in $H^1(\Omega)$ consisting of harmonic Steklov eigenfunctions. is described in Auchmuty [1,3]. It requires the solution of a sequence of constrained variational principles. The boundary traces of these eigenfunctions are L^2-orthogonal on the boundary and are proved to be bases of a scale of Hilbert spaces of functions on $\partial\Omega$. In Sections 4 and 6 various error estimates for Steklov approximations are obtained.

When the region is a planar disk, the Steklov eigenfunctions are the usual harmonic functions $r^m \cos m\theta$, $r^m \sin m\theta$ of Fourier analysis and the question of the approximation of harmonic functions on the unit disc by harmonic polynomials has a huge literature. The text of Axler, Bourdon and Ramey [4] is a recent introduction to the theory.

Here attention will be on the case where the region is a rectangle. In this case, the Steklov eigenfunctions are known explicitly see Auchmuty and Cho [5] or Girouard and Polterovich [6] where a completeness proof for this family is given. Computational results for Steklov approximations of certain harmonic functions regarded as solutions of Laplace's equations with various boundary value conditions are described in Sections 5 and 7. Dirichlet problems are considered in Sections 4 and 5 while results for Robin and Neumann problems are described in Sections 6 and 7.

For general regions, the Steklov eigenvalues and eigenfunctions are not (yet) known explicitly. However a number of authors have studied the numerical determination of these eigenfunctions including Cheng, Lin and Zhang [7], and Kloucek, Sorensen and Wightman [8]. The software FreeFem++ [9] has subroutines for the computation of Steklov eigenfunctions and eigenvalues that was used for confirmation of some of the analytical results described here.

Our general conclusion is that many harmonic functions are well-approximated by Steklov expansions with a relatively small number of Steklov eigenfunctions. They appear to provide very good approximations in the interior of the region and become quite oscillatory close to, and on, the boundary. It should be noted that this analysis extends to the solution of more general self-adjoint second order elliptic equations of the form $\mathcal{L}u = 0$ using similar general constructions as described in the paper [10].

2. Assumptions and notation

This paper treats various Laplacian boundary value problems on regions Ω in the plane $\mathbb{R}^2$. A region is a non-empty, connected, open subset of $\mathbb{R}^2$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. Some regularity of the boundary $\partial\Omega$ is required. Each component (= maximal connected closed subset) of the boundary is assumed to be a Lipschitz continuous closed curve. Let σ denote arc-length along a curve so the unit outward normal $\nu(z)$ is defined σ *a.e.*

$L^p(\Omega)$ and $L^p(\partial\Omega, d\sigma)$, $1 \le p \le \infty$ are the usual spaces with p-norm denoted by $\|u\|_p$ or $\|u\|_{p,\partial\Omega}$ respectively. When $p = 2$ these are real Hilbert spaces with inner products defined by

$$\langle u, v \rangle := \int_\Omega u \, v \; dxdy \quad \text{and} \quad \langle u, v \rangle_{\partial\Omega} := |\partial\Omega|^{-1} \int_{\partial\Omega} u \, v \; d\sigma.$$

$C(\overline{\Omega})$ is the space of continuous functions on the closure $\overline{\Omega}$ of Ω with the sup norm $\|u\|_b := sup_{\overline{\Omega}} |u(x, y)|$.

The weak jth derivative of u is $D_j u$ — and all derivatives will be taken in a weak sense. Then $\nabla u := (D_1 u, D_2 u)$ is the gradient of u and $H^1(\Omega)$ is the usual real Sobolev space of functions on Ω. It is a real Hilbert space under the standard H^1-inner product

$$[u, v]_1 := \int_\Omega [u \, v + \nabla u \cdot \nabla v] \; dxdy. \tag{2.1}$$

The corresponding norm is denoted $\|u\|_{1,2}$.

The region Ω is said to satisfy *Rellich's theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \le p < \infty$.

The boundary trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is the linear extension of the map restricting Lipschitz continuous functions on $\overline{\Omega}$ to $\partial\Omega$. The region Ω is said to satisfy a *compact trace theorem* provided the boundary trace mapping $\gamma : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ is compact. Theorem 1.5.1.10 of Grisvard [11] proves an inequality that implies the compact trace theorem for bounded regions in $\mathbb{R}^N$ with Lipschitz boundaries. Usually γ is omitted so u is used in place of $\gamma(u)$ for the trace of a function on $\partial\Omega$.

The *Gauss–Green* theorem holds on Ω provided

$$\int_\Omega u D_j v \; dxdy = \int_{\partial\Omega} \gamma(u) \, \gamma(v) \, \nu_j \; d\sigma \; - \int_\Omega v \, D_j u \; dxdy \quad \text{for } 1 \le j \le N \tag{2.2}$$

for all u, v in $H^1(\Omega)$. The requirements on the region will be