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A singularly perturbed convection–diffusion problem with a moving pulse

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ABSTRACT

A singularly perturbed parabolic equation of convection–diffusion type is examined. Initially the solution approximates a concentrated source. This causes an interior layer to form within the domain for all future times. Using a suitable transformation, a layer adapted mesh is constructed to track the movement of the centre of the interior layer. A parameter-uniform numerical method is then defined, by combining the backward Euler method and a simple upwinded finite difference operator with this layer-adapted mesh. Numerical results are presented to illustrate the theoretical error bounds established.

1. Introduction

Singularly perturbed convection–diffusion parabolic problems can be viewed as simple mathematical models of some pollutant being transported through a fast flowing medium. In this paper, we consider a model, where the width of the initial profile of the pollutant concentration approximates a point source.

In the case of smooth data, boundary layers can appear in the solutions of singularly perturbed parabolic problems. Globally accurate parameter-uniform numerical approximations [1] to the solutions of these kinds of problems can be constructed using layer-adapted meshes such as piecewise-uniform Shishkin meshes [2]. In the case of non-smooth data, additional interior layers can appear in the solution [3]. If the initial condition is discontinuous, then parameter-uniform globally accurate numerical methods do not exist for such problems [4,5].

In [6] we constructed and analysed a numerical method for a problem with a regularized step-function as the initial profile. In the current paper, the methodology (in both the construction and the associated theoretical analysis) is extended to a singularly perturbed convection–diffusion parabolic problem with a regularized delta-function as the initial profile.

The width of the layer in the regularized initial condition in [6] was directly related to the scale of the singular perturbation parameter ε contained within the differential equation. In the current paper, we examine the case of a layer in the initial condition having a potentially different scale to, what we shall call, the *normal scale* of $\mathcal{O}(\sqrt{\varepsilon})$. The normal scale is the scale of an interior layer emanating from a singularly perturbed parabolic equation of the form $-\varepsilon z_{yy} + z_t = 0$ [4,5], where the interior layer is moving (in time) along a direction orthogonal to the y-coordinate axis. Here we examine an initial condition involving a Gaussian profile with a standard deviation determined by two parameters ε and θ, where the value of θ determines how far the width of the pulse deviates from the normal layer width. We see that if the scale of the

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layer in the initial condition (of order $\mathcal{O}(\sqrt{\varepsilon/\theta})$) is significantly thinner than the normal scale ($\theta \gg 1$), then large gradients in time are observed initially and the magnitude of the approximation errors is adversely affected by the presence of this excessively thin pulse. In addition, we also consider the intermediate case of $C\varepsilon \ll \theta \ll C$, which lies between the case of no layer ($\theta \leq C\varepsilon$) and the case of a normal layer width ($\theta = C$) in the initial pulse. In this paper, we highlight how the error constants (in the theoretical error bounds) depend on this parameter θ. The asymptotic error bound given in Theorem 7, indicates a degradation in the error bound for the case of $\theta \neq \mathcal{O}(1)$. Note that, for any fixed value of the parameter θ, the numerical method is parameter-uniformly convergent with respect to the singular perturbation parameter ε, present in the differential equation.

In Section 2 we state the problem class examined in this paper and global parameter-explicit bounds on the solution are established. A transformation of the domain is introduced in Section 3, which is used to align the mesh with the trajectory of the interior layer. Sharper pointwise bounds on the partial derivatives of the solution are derived in Section 4, using a decomposition of the solution into a sum of regular, boundary layer and interior layer components. The numerical scheme is constructed in Section 5 and theoretical error bounds are established in Section 6. Some numerical results are presented and discussed in Section 7.

Notation: In this paper C denotes a generic constant that is independent of the parameter ε and the mesh parameters N and M. For any function z, we set $\|z\|_{\bar{G}} := \max_{(s,t)\in\bar{G}} |z(s,t)|$.

2. Continuous problem

Consider the following singularly perturbed parabolic problem: Find $\hat{u}$ such that

$$\hat{\mathcal{L}}_\varepsilon \hat{u} = \hat{f}(s,t), \quad (s,t) \in Q := (-1,1)\times(0,T], \quad \text{where } \hat{\mathcal{L}}_\varepsilon \hat{u} := -\varepsilon \hat{u}_{ss} + \hat{a}(t)\hat{u}_s + \hat{b}(s,t)\hat{u} + \hat{c}(t)\hat{u}_t, \tag{1a}$$

$$\hat{u}(s,0) = \phi(s;\varepsilon), \quad -1 \leq s \leq 1, \tag{1b}$$

$$\hat{u}(-1,t) = \phi_L(t), \qquad \hat{u}(1,t) = \phi_R(t), \quad 0 < t \leq T, \tag{1c}$$

$$\hat{a}(t) > \alpha > 0, \qquad \hat{c}(t) \geq c_0 > 0, \tag{1d}$$

$$\hat{b}(s,t) \geq \beta \geq 0, \qquad \hat{b}(s,t) + 2\hat{c}'(t) > 0, \quad (s,t) \in Q. \tag{1e}$$

Note that, by using the standard transformation of $\hat{u} = \hat{v}e^{\gamma t}$, we see that (1e) is a mild constraint on the data.

The initial condition ϕ is smooth, but has an ε-dependent Gaussian profile in the vicinity of $s = 0$. The initial condition is assumed to be of the form

$$\phi(s,\varepsilon) = g_1(s) + g_2(s)e^{-\theta\frac{s^2}{\varepsilon}}, \quad \theta > C\varepsilon; \tag{1f}$$

where $g_1(s), g_2(s)$ are smooth functions with the additional compatibility assumptions of

$$g_2^{(i)}(-1) = g_2^{(i)}(1) = 0, \quad i = 0, 1, 2. \tag{1g}$$

These additional assumptions ensure that the pulse $e^{-\theta\frac{s^2}{\varepsilon}}$ has no influence on the smoothness of the solution at the end-points $(-1,0)$ and $(1,0)$.

The case where $0 < \theta \leq C\varepsilon$ is not of interest to us here, as in this case no interior layer will form in the solution. Observe that as $\theta/\varepsilon \to \infty$ the width of the pulse narrows and the pulse can be viewed as a regularized delta function. We limit our investigation of the effect of θ, by restricting the parameter to the case of $\theta = \mathcal{O}(1)$, so that we assume that

$$0 < C_* \leq \theta \quad \text{and} \quad \frac{\theta T}{c_0} \leq C^*. \tag{1h}$$

The error constants C in our final error bound do depend on the constant C^*. By assuming that $\frac{\theta T}{c_0} \leq C^*$, we can then utilize the transformation $\hat{u} = \hat{v}e^{\frac{2\theta t}{c_0}}$ so that there is no further loss in generality in assuming the constraint

$$\hat{b} \geq 2\theta > 0, \quad (s,t) \in Q.$$

The characteristic curve associated with the reduced differential equation (formally set $\varepsilon = 0$ in (1a)) can be described by the set of points

$$\Gamma^* := \left\{(d(t),t) \,|\, d'(t) = \frac{\hat{a}(t)}{\hat{c}(t)},\ d(0) = 0\right\}.$$

Note that $d'(t) > 0$, which implies that the centre of the pulse moves rightwards with time.

We also define the two subdomains of Q either side of Γ^* by

$$Q^- := \{(s,t) \in Q \,|\, s < d(t) < 1\} \quad \text{and} \quad Q^+ := \{(s,t) \in Q \,|\, s > d(t) > -1\}.$$