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A posteriori error analysis of nonconforming finite element methods for convection–diffusion problems[☆]

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ABSTRACT

A unified framework is established for the a posteriori error analysis of nonconforming finite element approximations to convection–diffusion problems. Under some certain conditions, the theory assures the semi-robustness of residual error estimates in the usual energy norm and the robustness in a modified norm, and applies to several nonconforming finite elements, such as the Crouzeix–Raviart triangular element, the nonconforming rotated (NR) parallelogram element of Rannacher and Turek, the constrained NR parallelogram element, etc. Based on the general error decomposition in different norms, we show that the key ingredients of error estimation are the existence of a bounded linear operator $\Pi : V_h^c \to V_h^{nc}$ with some elementary properties and the estimation on the consistency error related to the particular numerical scheme. The numerical results are presented to illustrate the practical behavior of the error estimator and check the theoretical predictions.

1. Introduction

Nonconforming finite element methods are of considerable interest in the numerical approximation of partial differential equations, for they easily fulfill the discrete version of the Babuška–Brezzi condition and enjoy better stability properties compared to the conforming finite elements. Moreover, the unknowns are localized at the element faces so that each degree of freedom belongs to at most two elements, which results in a cheap local communication for a parallelization. Therefore, in the recent decade the nonconforming elements have been considered and analyzed on the convergence properties for solving the convection–diffusion problems. Specifically, John et al. [1] first proposed a class of streamline-diffusion finite element methods based on nonconforming triangular elements and gave an analysis of stability and convergence. Therein, the error analysis relies on the fact that the nonconforming finite element space has a conforming subspace and thus cannot be applied to the nonconforming quadrilateral elements, such as the rotated $\mathbb{Q}_1$ element of Rannacher and Turek [2], where the space of continuous piecewise bilinear functions is not a subspace. Later, John et al. [3] overcame this difficulty and allowed one to handle both triangular and quadrilateral meshes. In one word, theoretical and numerical investigations have shown that by adding certain jump terms to the bilinear form the known $\mathcal{O}(h^{3/2})$ order in the streamline-diffusion norm can

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be preserved as in the conforming case for the lowest-order elements. We refer the readers to [4–8] for further survey and analysis on the nonconforming streamline-diffusion methods. For other nonconforming methods with subgrid viscosity, face and interior penalty, see [9–11].

For various stabilized finite element methods based on the conforming elements, the a posteriori error analysis has reached a certain level of maturity, see [12–19] and the references therein. By contrast, the a posteriori error analysis of nonconforming finite element approximations to convection–diffusion equations is a much less explored topic. El Alaoui et al. [11] derived a semi-robust error estimator for nonconforming finite element discretization with face penalty for convection–diffusion equations, where a factor appears in the lower error bound and scales at most as the square root of the Péclet number. In [20] the semi-robust a posteriori error estimation for the nonconforming streamline-diffusion methods [1] was presented, and the error analysis was also shown to be applied to other nonconforming finite element methods with face penalty [11] and subgrid viscosity [10]. In our latest paper [21], based on the previous works a unified framework was established in order to derive a posteriori error estimates for various conforming and nonconforming finite element methods for convection–diffusion problems. The error estimator is shown to be robust with respect to the diffusion coefficient in a modified norm which incorporates a discrete energy norm, a discrete dual semi-norm of the convective derivative and jumps of the approximate solution over element faces. Based on a general error decomposition, we show that the key ingredient of error estimation is the estimation on the consistency error related to the particular numerical scheme, and the remaining terms can be bounded in a unified way. The works mentioned above only focus on the nonconforming triangular elements and rely on the existence of a conforming subspace.

When considering nonconforming quadrilateral elements such as the rotated $\mathbb{Q}_1$ element, the same difficulty mentioned above appears in the a posteriori error estimation. For the pure diffusion problem, a bounded linear operator $\Pi : V_h^{\mathrm{c}} \to V_h^{\mathrm{nc}}$ with some elementary properties was developed by Carstensen et al. [22] to handle the nonconforming quadrilateral case. In this paper, we verify more properties of the operator Π (see (2.8) and Lemma 2.1) which are closely related to the convection–diffusion problem, and derive the a posteriori error estimates applied to the nonconforming quadrilateral elements.

The aim of this paper is to extend the error analysis in [21] to the nonconforming quadrilateral case and establish a unified framework for the a posteriori error analysis of nonconforming finite element approximations to convection–diffusion problems. Under some certain conditions (see the hypotheses (H1)–(H2) and (3.7)), the theory assures the semi-robustness of residual error estimates in the usual energy norm and the robustness in the modified norm from [21], and applies to several nonconforming finite elements, such as the Crouzeix–Raviart triangular element, the nonconforming rotated (NR) parallelogram element of Rannacher and Turek, the constrained NR parallelogram element, etc. In fact, the constrained NR parallelogram element of Hu and Shi is equivalent to the double set parameter element presented in [23]. Based on the general error decomposition in different norms, we show that the key ingredients of error estimation are the existence of a bounded linear operator $\Pi : V_h^{\mathrm{c}} \to V_h^{\mathrm{nc}}$ with some elementary properties and the estimation on the consistency error related to the particular numerical scheme, see Remark 4.1.

The remaining remainder of this paper is organized as follows. Section 2 displays some notations and preliminaries, such as the model problem, the conforming and nonconforming finite element spaces with the hypotheses (H1)–(H2) and some interpolation error estimates, and a posteriori error estimator. In Section 3, semi-robust a posteriori error estimates are presented in an abstract framework based on a general error decomposition in the usual energy norm. In Section 4, robust a posteriori error estimates are presented in an abstract framework based on a general error decomposition in the modified norm. Application of the theory to the nonconforming finite element methods is given in Section 5. Therein, we verify the hypotheses (H1)–(H2) for the nonconforming rotated $\mathbb{Q}_1$ element, and then bound the consistency error satisfying the condition (3.7) for the corresponding nonconforming streamline-diffusion method proposed in [3]. Finally, in Section 6 several numerical experiments are presented in order to check the theoretical predictions.

2. Notations and preliminaries

2.1. Model problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\partial\Omega$. For any given open subset S of Ω, $(\cdot,\cdot)_S$ and $\|\cdot\|_S$ denote the usual integral inner product and the corresponding norm of both $L^2(S)$ and $[L^2(S)]^2$, respectively. If $S = \Omega$, the subscript will be omitted.

We consider the convection–diffusion problem with homogeneous boundary condition

$$\begin{cases} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f, & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$, $\mathbf{b} \in [\mathcal{C}^{0,\frac{1}{2}}(\overline{\Omega})]^2$ and $c \in L^\infty(\Omega)$ are the diffusion coefficient, the velocity field and the reaction coefficient, respectively. Without loss of generality, we assume that (2.1) is non-dimensionalized so that $\|\mathbf{b}\|_{[L^\infty(\Omega)]^2}$ and the length scale of Ω are of order unity; hence, the parameter ε is the reciprocal of the Péclet number.

The standard weak formulation of the problem (2.1) reads: find $u \in H_0^1(\Omega)$ such that

$$\varepsilon(\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u + cu, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{2.2}$$

Under the assumption that $c - \frac{1}{2}\mathrm{div}\mathbf{b} \geq c_0 > 0$, there is a unique solution of (2.2).