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Option pricing with Legendre polynomials

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ABSTRACT

Here we develop an option pricing method based on Legendre series expansion of the density function. The key insight, relying on the close relation of the characteristic function with the series coefficients, allows to recover the density function rapidly and accurately. Based on this representation for the density function, approximations formulas for pricing European type options are derived. To obtain highly accurate result for European call option, the implementation involves integrating high degree Legendre polynomials against exponential function. Some numerical instabilities arise because of serious subtractive cancellations in its formulation (96) in Proposition A.1. To overcome this difficulty, we rewrite this quantity as solution of a second-order linear difference equation and solve it using a robust and stable algorithm from Olver. Derivation of the pricing method has been accompanied by an error analysis. Errors bounds have been derived and the study relies more on smoothness properties which are not provided by the payoff functions, but rather by the density function of the underlying stochastic models. This is particularly relevant for options pricing where the payoffs of the contract are generally not smooth functions. The numerical experiments on a class of models widely used in quantitative finance show exponential convergence.

1. Introduction

In option pricing, Feynman–Kac formula [1] establishes a link between the conditional expectation of the value of a contract payoff function under the risk-neutral measure and the solution of a partial differential equation. In the research areas covered by this theorem, various numerical pricing techniques can be developed. Existing numerical methods can be classified into three major groups: partial integro-differential equation methods, Monte Carlo simulations and numerical integration methods. Each of them has its advantages and disadvantages for different financial models and specific applications. In this paper, we concentrate on the last group for the pricing of European type option.

The point-of-departure for pricing European option with numerical integration techniques is the risk-neutral valuation formula:

$$V(x, t_0 = 0) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[V(S_T, T)|S_0 = x] = e^{-rT}\int_{\mathbb{R}} V(y, T)\tilde{f}(y|x)dy \qquad (1)$$

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with $\mathbb{E}_{\mathbb{Q}}$ the expectation operator under risk-neutral measure $\mathbb{Q}$, S_t the underlying asset price at t and T the option maturity. $V(x, t)$ denotes the option value at t with x the state variable. $\tilde{f}(y|x)$ is the probability density function of S_T given $S_0 = x$ and r the risk-free interest rate.

Unfortunately, for many relevant pricing processes, their probability densities are usually unknown. On the other hand, the Fourier transform of these densities, i.e., the characteristic functions, are often available. For instance, from the Levy–Khintchine theorem [2] the characteristic functions of Levy processes are known. Or characteristic functions have been derived in the pure diffusion context with stochastic volatility [3] and with stochastic interest rates [4]. Hence, the Fourier transform methods for option pricing have been naturally considered by many authors (see [5] and references therein). Subsequently, some new numerical methods are proposed. For example, The quadrature method (QUAD) method was introduced by Andricopoulos et al. [6], the Convolution method (CONV) was presented by Lord et al. [7]. A fast Hilbert transform approach was considered by Feng and Linetsky [8]. The highly efficient Fourier-cosine series (COS) technique, based on Fourier-cosine series expansion of the density function, was proposed by Fang and Oosterlee [9] and has generated other developments by Hurn et al. [10] or by Ding et al. [11]. Recently, Necula et al. [12] have employed the modified Gram–Charlier series expansion, known as the Gauss–Hermite expansion, for the density function and obtained a closed form pricing formula for European option.

In this manuscript, we consider an alternative and propose to expand the probability density function $\tilde{f}(y)$, restricted on a finite interval $[a, b]$, using Legendre polynomials when the characteristic function is known. For approximating non periodic function on a finite interval, among the class of basis functions, it is usually recommended to use either Legendre polynomials or Chebyshev polynomials (see page 510 Table A.1 in [13]). Legendre polynomial offers tractability property allowing to compute analytically many quantities of interests. For example, Legendre polynomial has an analytical formula for its Fourier transform as in (15), which is instrumental and used to recover the coefficients A_n in the series expansion of the density function (26). The Fourier transform for Chebyshev polynomials does not have a simple closed form and requires some numerical approximations (see discussion in [14]). Moreover the experiments show this formula is numerically stable for large n. Generally, the classical Legendre series offers the simplest method of representing a function using polynomial expansion means [15]. Also the recent analysis by Cohen and Tan [16] shows Legendre polynomial approximation yields an error at least an order of magnitude smaller than the analogous Taylor series approximation and the authors strongly suggest that Legendre expansions, instead of Taylor expansions, should be used when global accuracy is important. Finally, polynomials are convenient to manipulate in general and we compute simply the European option pricing formula by integrating the payoff against Legendre polynomial functions.

Adrien Marie Legendre, a French mathematician who discovered the famous polynomials, was never aware of that how much it will be used in developing mathematics. This Legendre polynomial is being used by mathematicians and engineers for variety of mathematical and numerical solutions. For example, in physics, Legendre and Associate Legendre polynomials are widely used in the determination of wave functions of electrons in the orbits of an atom [17,18] and in the determination of potential functions in the spherically symmetric geometry [19]. In numerical analysis, Legendre polynomials are used to efficiently calculate numerical integrations by Gaussian quadrature method [20].

Legendre polynomials is not widely used in quantitative finance but not new. For example, Pulch et al. [21] consider the fair price of options as the expected value of a random field where the input volatility parameter is written as a linear function of uniform random variable. The Polynomial chaos theory using Legendre polynomial yields an efficient approach for calculating the required fair price. Or in [22], the authors develop arbitrage free interest rate models for a family of term structures parametrized by linear combinations of Legendre polynomials. Each polynomial provides a clear interpretation in terms of the type of movements that they generate for the term structure (see also [23,24]).

To our knowledge, it is the first time that Legendre polynomials are used to expand the probability density function of asset prices and option pricing. To recover rapidly and accurately the density function, our key insight relies on the close relation of the characteristic function with the series coefficients of the Legendre polynomials expansion of the density function (see our result in Theorem 2.1). Based on this representation for the density function, approximations formulas for pricing European type options are derived. To obtain highly accurate result for European call option, the implementation involves integrating high degree Legendre polynomials against exponential function. Some numerical instabilities arise because of serious subtractive cancellations in its formulation (96) in Proposition A.1. To overcome this difficulty, we rewrite this quantity as solution of a second-order linear difference equation in Proposition 3.3. To solve this equation in a stable way, we use Olver's algorithm which allows to evaluate these quantities to machine accuracy. Then we develop an analysis to provide estimations of the errors. We believe that a rigorous error estimate is of first importance because the accuracy of our expansion formulas depends on the regularity of the density function. Once done, it brings confidence in the derived expansion and sheds light on the needed assumptions (see our results in Propositions 4.2, 4.3 and 4.5).

This paper is structured as follows. In Section 2, we develop the series expansion of the density function using Legendre polynomials. Based on this, we derive, in Section 3, the formulas for pricing European type options and propose a robust and stable procedure for the implementation. An error analysis is presented in Section 4. Some numerical experiments are given in Section 5. The final section concludes.

2. Series expansion of density function with Legendre polynomials

The objective is to estimate the density function $\tilde{f}(y)$ using Legendre polynomials given its characteristic function.