Journal of Computational and Applied Mathematics 326 (2017) 204–216

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A splitting iterative method for solving second kind integral equations in reproducing kernel spaces

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ARTICLE INFO

Article history:
Received 5 November 2016
Received in revised form 11 April 2017

Keywords:
Iterative methods
Integral equation of second kind
Reproducing kernel theory
Reproducing kernel space

ABSTRACT

In the present paper, we propose a new iterative method to solve integral equations of the second kind in reproducing kernel Hilbert spaces (RKHS). At first, we make appropriate splitting in second kind integral equations and according to this splitting the iterative method will be constructed; then, bases of RKHS and reproducing kernel spaces properties are used to convert this problem to linear system of equations. We move between reproducing kernel spaces by changing bases in order to achieve more accurate approximate solutions. Classically, in iterative RKHS method, the number of iterations should be the same as the number of points; here, we present a type of iterative RKHS method without this limitation. Convergence of the proposed method is investigated, and the efficiency of the method is demonstrated through various examples.

1. Introduction

In this paper, we consider integral equations of the second kind whether Fredholm or Volterra

$$u(x) + \lambda(x) \int_0^1 K(x, y)u(y)dy = f(x) \tag{1.1}$$

where, for all $x \in [0, 1]$, $\lambda(x) \neq 0$ and λ, K, f are known smooth functions, and u is the solution of Eq. (1.1) to be determined. Without loss of generality in (1.1), we could consider

$$K(x, y) = \begin{cases} K(x, y) & x \geq y \\ 0 & x < y. \end{cases}$$

Equivalently, in the operator form, we have

$$Lu(.) = f(.), \tag{1.2}$$

in which

$$Lu(.) := u(.) + \lambda(.) \int_0^1 K(., y)u(y)dy, \tag{1.3}$$

and we assume that problem has a unique solution. Integral equations are subject of many papers; here, we give a brief review of some papers which study integral equations with reproducing kernel methods. In [1], Du et al. apply the RKHS

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http://dx.doi.org/10.1016/j.cam.2017.05.025

method on Fredholm integral equations of the first kind, and they investigated stability of this ill-posed problem in RKHS. In [2], Jiang et al. studied the integral equation of the third or first kind in $W_2^1[0, 1]$, by use of the RKHS method. In [3], Chen et al. investigate Hilbert type singular integral equation: first, a transformation is used to remove singularity and then the problem is solved by their improved RKHS method. In [4], Ketabchi et al. investigate the error of the RKHS method for solving second kind Volterra integral equations in $W_2^1[0, 1]$ and $W_2^2[0, 1]$. In [5], Babolian et al. expand [4] and consider error of the RKHS method for solving second kind Volterra integral equations in $W_2^m[0, 1]$.

In the last decade, several authors investigated application of Reproducing the kernel method for a wide variety of mathematical problems, such as differential equations, integral equations, inverse, probability and stochastic problems [6–10]. In most papers which use the RKHS method, authors have generally two point of views: first, they try to find solution of linear operator in a fix RKHS; second, if the operator is not linear, they split the operator into linear and nonlinear parts and then they use the standard iterative RKHS method to solve their problems in a fix RKHS. Although the standard iterative RKHS method is efficient, this method possesses limitations; for instance, in order to attain more accurate approximate solution, we should increase the number of base functions and then use the Gram–Schmidt process to make the bases orthonormal. In this paper, we split second kind integral equations into two parts; then, by use of Reproducing kernel bases and reproducing property of RKHS $W^m[0, 1]$, a second kind integral equation is converted into a system of linear equations. We try to remove the RKHS method limitations and attain a better or at least the equipotential method.

The remainder of this paper is organized as follows. In Section 2, we present definitions and some useful properties of reproducing kernel spaces. In Section 3, our method is implemented by the use of reproducing kernel space bases to solve the system of Fredholm integral equations. In Section 4, error estimate and convergence analysis of our method is investigated. Section 5 contains some numerical examples illustrating the application of the proposed method. We end the paper with some conclusions.

2. Preliminaries and notations

In this section, we briefly review reproducing kernel properties of the Hilbert space $W^m[a, b]$ and also fixing the notation used in this paper.

The Hilbert function space, $W^m[a, b]$, is defined as the linear function space

$$W^m[a, b] = \{f \,|\, f, f', \ldots, f^{(m-1)} \text{ are absolutely continuous}, f^{(m)} \in L^2[a, b]\},$$

which is equipped with the following inner product

$$\langle f, g\rangle_{W^m} = \sum_{i=0}^{m-1} f^{(i)}(a)g^{(i)}(a) + \int_a^b f^{(m)}(x)g^{(m)}(x)dx. \tag{2.1}$$

The inner product (2.1) induces the following Hilbert norm:

$$\|f\|_{W^m} = \sqrt{\langle f, f\rangle_{W^m}}. \tag{2.2}$$

The following theorem presents an interesting property of the Hilbert space $W^m[a, b]$.

Theorem 2.1 ([9]). *The Hilbert function space $W^m[a, b]$ is a reproducing kernel space with the conjugate symmetric reproducing kernel $R_m(x, y)$ given by*

$$R_m(x, y) = \begin{cases} lR_m(x, y) = \sum_{i=1}^{2m} c_i(y)x^{i-1} & x < y \\ rR_m(x, y) = \sum_{i=1}^{2m} d_i(y)x^{i-1} & x \ge y \end{cases} \tag{2.3}$$

in which coefficients $c_i(y)$, $d_i(y)$ are the solutions of the following system of differential equations

$$\begin{cases} (-1)^m \dfrac{\partial^{2m} R(x, y)}{\partial x^{2m}} = \delta(x - y), \\ \dfrac{\partial^i R(a, y)}{\partial x^i} - (-1)^{m-i-1}\dfrac{\partial^{2m-i-1} R(a, y)}{\partial x^{2m-i-1}} = 0, \\ \dfrac{\partial^{2m-i-1} R(b, y)}{\partial x^{2m-i-1}} = 0, \quad i = 0, 1, \ldots, m-1. \end{cases} \tag{2.4}$$

In particular, each function $f \in W^m[a, b]$ satisfies the following reproducing property:

$$f(y) = \langle f(.), R_m(., y)\rangle_{W^m} \quad \forall y \in [a, b].$$