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Numerical analysis of a characteristic stabilized finite element method for the time-dependent Navier–Stokes equations with nonlinear slip boundary conditions[☆]

Feifei Jing[a], Jian Li[b,c], Zhangxin Chen[a,d,*], Zhonghua Zhang[e]

[a] College of Mathematics and Statistics, Center for Computational Geoscience, Xi'an Jiaotong University, Xi'an 710049, PR China
[b] Department of Mathematics, School of Arts and Sciences, Shaanxi University of Science and Technology, Xi'an 710021, PR China
[c] Institute of Computational Mathematics and its Applications, Baoji University of Arts and Sciences, Baoji 721013, PR China
[d] Department of Chemical and Petroleum Engineering, Schulich School of Engineering, University of Calgary, 2500 University Drive N. W. Calgary, Alberta, T2N 1N4, Canada
[e] School of Sciences, Xi'an University of Science and Technology, Xi'an 710054, PR China

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ABSTRACT

Based on a characteristic method, this work is concerned with a finite element approximation to the time-dependent Navier–Stokes equations with nonlinear slip boundary conditions. Since this slip boundary condition of friction type contains a subdifferential property, its continuous variational problem is formulated as an inequality, which can turn into an equality problem by using a powerful regularized method. Then a fully discrete characteristic scheme under the stabilized lower order finite element pairs is proposed for the equality problem. Optimal error estimates for velocity and pressure are derived under the corresponding L^2, H^1-norms. Finally, a smooth problem test is reported to demonstrate the theoretically predicted convergence order and the expected slip phenomena, and the simulation of a bifurcated blood flow model is displayed to illustrate the efficiency of the proposed method.

1. Introduction

In this paper, we will consider the mathematical model of viscous incompressible fluid, which can be written as the following time-dependent Navier–Stokes equations

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in} \quad \Omega \times J, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times J, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega \times \{0\}, \end{cases} \tag{1.1}$$

where $J = (0, T]$ $(0 < T < \infty)$ is a given time interval, $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with a Lipschitz continuous boundary $\Gamma = \partial\Omega$, $\boldsymbol{u}(x, t)$ and $\boldsymbol{f}(x, t)$ denote the flow velocity and the external force, $p(x, t)$ is pressure, and $\nu > 0$ is the

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* Corresponding author at: Department of Chemical and Petroleum Engineering, Schulich School of Engineering, University of Calgary, 2500 University Drive N. W. Calgary, Alberta, T2N 1N4, Canada.

E-mail addresses: jingfei.cool@163.com (F. Jing), jiaaanli@gmail.com (J. Li), zhachen@ucalgary.ca (Z. Chen), wwwzhonghua@sohu.com (Z. Zhang).

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kinematic viscosity. Moreover, the boundary conditions are presented as follows:

$$\begin{cases} \boldsymbol{u} = 0 & \text{on} \quad \Gamma_D, \\ u_{\boldsymbol{n}} = 0, \quad |\boldsymbol{\sigma_\tau}| \le g, \quad \boldsymbol{\sigma_\tau u_\tau} + g|\boldsymbol{u_\tau}| = 0 & \text{on} \quad \Gamma_S, \end{cases} \tag{1.2}$$

where $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_S$. The adhesive Dirichlet boundary condition is imposed on Γ_D, and for Γ_S, a nonlinear slip and non-leak boundary conditions are considered. Assume that both Γ_S and Γ_D are not empty and $\Gamma_D \cap \Gamma_S = \emptyset$. Here and what follows, the unit outward normal vector and the tangent vector to the boundary are denoted by $\boldsymbol{n}$ and $\boldsymbol{\tau}$, respectively. For a vector field $\boldsymbol{v}$ on the boundary, $\boldsymbol{v} \cdot \boldsymbol{n}$ and $\boldsymbol{v} \cdot \boldsymbol{\tau}$ are its normal and tangential components. Let $v_{\boldsymbol{n}} \equiv \boldsymbol{v} \cdot \boldsymbol{n}$ and $\boldsymbol{v_\tau} \equiv \boldsymbol{v} - v_{\boldsymbol{n}}\boldsymbol{n}$. Denote by $\boldsymbol{\sigma_\tau}(\boldsymbol{u}) = \nu \frac{\partial \boldsymbol{u_\tau}}{\partial \boldsymbol{n}}$, independent of p, the tangential component of stress vector defined on Γ_S. The frictional function g, is assumed to be continuous on $\overline{\Gamma}_S$ and strictly positive on Γ_S. This friction type of boundary conditions was first introduced by Fujita in [1] and appeared in the modeling of blood flow in a vein of an arterial sclerosis patient and some other models.

For such boundary conditions (1.2), Fujita in [2] showed existence and uniqueness of a weak solution for the Stokes problem. From a theoretical point, some well-posedness analyses for the Stokes problem with nonlinear slip boundary conditions have been discussed during the past years [3–6]. In addition, numerical results for the steady Stokes and Navier–Stokes problems with such boundary conditions can be found in [7–12]. However, to our knowledge, there has not been much work on an analysis of finite element (FE) approximations to the unsteady problems with such boundary conditions. Djoko in [13] considered a semi-discrete scheme in time for the unsteady Stokes variational inequality problem by means of a regularized method, Kashiwabara in [14] investigated the weak solutions of the continuous variational inequality problem governed by the non-stationary Navier–Stokes equations, Li in [15] used the regularized method to obtain existence, uniqueness and regularity of global weak solutions to the two-dimensional time-dependent Navier–Stokes equations with nonlinear slip boundary conditions, and also stabilized FE methods are employed to solve this problem in [16], where the following convergence estimates with respect to a regularization parameter ε are established:

$$\begin{cases} \|\boldsymbol{u} - \boldsymbol{u}^\varepsilon\|_{L^\infty(0,T;\boldsymbol{L}^2)} \le C\varepsilon^{\frac{1}{2}}, \\ \|\boldsymbol{u} - \boldsymbol{u}^\varepsilon\|_{L^2(0,T;\boldsymbol{H}^1)} + \|p - p^\varepsilon\|_{L^2(0,T;L^2)} \le C\varepsilon^{\frac{1}{2}}, \end{cases}$$

where $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ are the solutions of the Navier–Stokes type variational inequality problem and its regularized problem, respectively, and the constant $C > 0$ is independent of ε.

Since the characteristic methods can effectively weaken a non-physical phenomenon caused by nonlinear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ [17,18], which via rewriting the governing equations (1.1) in terms of Lagrangian coordinates defined by the particle trajectories associated with the problem under consideration [19]. The Lagrangian treatment can greatly reduce a time truncation error in the Eulerian method [20], and the characteristic methods have been shown to possess remarkable stability properties [21,22]. Furthermore, it is well-known that a regularized method plays a key role in theoretical and numerical analysis of a variational inequality problem, which turns the variational inequality into equations. In this work, with the help of regularized technology, we combine the characteristic method with the pressure projection stabilized method to solve the time-dependent Navier–Stokes problem with nonlinear slip boundary conditions, and we derive the optimal error estimates based on the following FE approximation:

$$\begin{cases} \|\boldsymbol{u}^\varepsilon(t_m) - \boldsymbol{u}_h^\varepsilon(t_m)\|_{H^1} + \|p^\varepsilon(t_m) - p_h^\varepsilon(t_m)\|_{L^2} \le Ch, \\ \|\boldsymbol{u}^\varepsilon(t_m) - \boldsymbol{u}_h^\varepsilon(t_m)\|_{L^2} \le Ch^2. \end{cases}$$

The organization of this paper is given as follows. In the next section, we will introduce some function spaces, existence of weak solutions of the discussed problem, the characteristic method and the corresponding regularized problem. In Section 3, we will develop a first-order fully discrete scheme of the characteristic regularized continuous equality problem. Our analysis shows that this fully discrete scheme is unconditionally stable provided that the characteristics are transported by a divergence-free velocity field. Optimal error estimates for the characteristic stabilized method are derived in Section 4. This work ends with a section of numerical examples. Slip and non-slip phenomena are shown that depend on the friction function, the obtained optimal error estimates are consistent with theoretical analysis, results of the blood flow model further illustrate the feasibility of the proposed method.

2. Statement of the Navier–Stokes equations with nonlinear slip boundary conditions

2.1. Weak form of the problem

Let $H_0^1(\Omega)$ be the standard Sobolev space [23] equipped with the usual norm $\|\cdot\|_1$. For function spaces corresponding to velocity and pressure, we introduce closed subspaces of $[H^1(\Omega)]^2$ or $L^2(\Omega)$ as follows:

$$\boldsymbol{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^2 : \boldsymbol{v}|_{\Gamma_D} = 0, v_{\boldsymbol{n}}|_{\Gamma_S} = 0\}, \boldsymbol{V}_o = \{\boldsymbol{v} \in \boldsymbol{V} : \operatorname{div} \boldsymbol{v} = 0\}, \boldsymbol{Y} = \boldsymbol{L}^2(\Omega), \mathring{\boldsymbol{V}} = [H_0^1(\Omega)]^2,$$

$$Q = L_0^2(\Omega) = \left\{ q \in L^2(\Omega), \int_\Omega q \, d\boldsymbol{x} = 0 \right\}, \boldsymbol{H} = \{\boldsymbol{v} \in L^2(\Omega)^2 | \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } v_{\boldsymbol{n}} = 0 \text{ on } \partial\Omega\}.$$