Journal of Computational and Applied Mathematics 320 (2017) 221–241

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A non-linear structure-preserving matrix method for the computation of the coefficients of an approximate greatest common divisor of two Bernstein polynomials

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HIGHLIGHTS

- An approximate greatest common divisor of two Bernstein polynomials is computed.
- An approximate polynomial factorisation and the Sylvester matrix are used.
- The results from the two methods are very similar.

ARTICLE INFO

Article history:
Received 21 March 2016
Received in revised form 29 November 2016

Keywords:
Approximate greatest common divisor
Sylvester resultant matrix
Structure-preserving matrix methods

ABSTRACT

This paper describes a non-linear structure-preserving matrix method for the computation of the coefficients of an approximate greatest common divisor (AGCD) of degree t of two Bernstein polynomials $f(y)$ and $g(y)$. This method is applied to a modified form $S_t(f,g)Q_t$ of the tth subresultant matrix $S_t(f,g)$ of the Sylvester resultant matrix $S(f,g)$ of $f(y)$ and $g(y)$, where Q_t is a diagonal matrix of combinatorial terms. This modified subresultant matrix has significant computational advantages with respect to the standard subresultant matrix $S_t(f,g)$, and it yields better results for AGCD computations. It is shown that $f(y)$ and $g(y)$ must be processed by three operations before $S_t(f,g)Q_t$ is formed, and the consequence of these operations is the introduction of two parameters, α and θ, such that the entries of $S_t(f,g)Q_t$ are non-linear functions of α, θ and the coefficients of $f(y)$ and $g(y)$. The values of α and θ are optimised, and it is shown that these optimal values allow an AGCD that has a small error, and a structured low rank approximation of $S(f,g)$, to be computed.

1. Introduction

The need to calculate the points of intersection of two polynomial curves $p(x,y)=0$ and $q(x,y)=0$ arises frequently in computer aided geometric design (CAGD), and an important part of this calculation is the computation of the greatest common divisor (GCD) of $p(x,y)$ and $q(x,y)$. Resultant matrices are frequently used for this computation, and these matrices and other polynomial computations also occur in robotics [1], computer vision [2], computational geometry, for example, the implicitization of parametric curves and surfaces [3] and the construction of surfaces [4,5], control theory [6] and the computation of multiple roots of a polynomial [7,8]. There are several resultant matrices, including the Sylvester, Bézout and companion resultant matrices, of which the Sylvester matrix is the most popular, presumably because its entries are linear,

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http://dx.doi.org/10.1016/j.cam.2017.01.035
0377-0427/

even though it is larger than the Bézout and companion matrices. This property of the entries of the Sylvester matrix must be compared with the entries of the Bézout and companion matrices, which are bilinear and non-linear, respectively [9].

There has been extensive work on the theoretical and numerical properties of resultant matrices for polynomials expressed in the power basis, but much less work has been performed on resultant matrices for polynomials expressed in the Bernstein basis, which is of particular interest in CAGD because of its widespread use in this application. Explicit forms for the entries of the Bézout resultant matrix [10], the companion resultant matrix [11] and the Sylvester resultant matrix [12] of the Bernstein polynomials $\hat{f}(y)$ and $\hat{g}(y)$,

$$\hat{f}(y) = \sum_{i=0}^{m} \hat{a}_i \binom{m}{i} (1-y)^{m-i} y^i \quad \text{and} \quad \hat{g}(y) = \sum_{i=0}^{n} \hat{b}_i \binom{n}{i} (1-y)^{n-i} y^i, \tag{1}$$

have been developed but there has been significantly less investigation into their numerical properties. These properties are worthy of consideration because these resultant matrices contain combinatorial terms, and thus even if the magnitude of each coefficient $\hat{a}_i$ and $\hat{b}_j$ is of order one, the entries of these matrices may span several orders of magnitude, which may cause numerical problems.

It was noted above that the computation of the points of intersection of two polynomial curves requires the GCD of their polynomial forms. It is necessary to distinguish between polynomials whose coefficients are, and are not, subject to error because the GCD is defined for exact polynomials only, but the coefficients of polynomials in practical problems are subject to error. Inexact (noisy) polynomials must therefore be considered, and this leads to an approximate greatest common divisor (AGCD) of two polynomials whose coefficients are subject to error. This paper considers, therefore, the computation of the coefficients of an AGCD of degree t of the noisy forms $f(y)$ and $g(y)$ of the exact polynomials $\hat{f}(y)$ and $\hat{g}(y)$,

$$f(y) = \sum_{i=0}^{m} a_i \binom{m}{i} (1-y)^{m-i} y^i \quad \text{and} \quad g(y) = \sum_{i=0}^{n} b_i \binom{n}{i} (1-y)^{n-i} y^i,$$

by applying the method of structured non-linear total least norm (SNTLN) [13] to a modified form $S_t(f,g)Q_t$ of the tth subresultant matrix $S_t(f,g)$, where Q_t is a diagonal matrix of combinatorial terms. The calculation of the degree t is considered in [14,15] and it is assumed this calculation has been performed, and the value of t is therefore known. The computation of the GCD of $\hat{f}(y)$ and $\hat{g}(y)$ by Euclid's algorithm is investigated in [16] and several stopping criteria for the termination of the divisions in finite precision arithmetic are considered. The results show that roundoff error can cause a serious deterioration in the computed GCD and that a numerically robust method is required for this computation.

One application of the work described in this paper is, as noted above, the calculation of the points of intersection of two curves. Another application is the computation of multiple roots of a polynomial, where the multiplicity of a root defines the smoothness of curves and surfaces at their intersection point. These roots are important for mechanical design because the stresses in sharp corners of an object may become very large, and much larger than in its interior, such that the object may fracture when in operation. These high stress levels can be reduced substantially by rounding off the corners of the intersecting curves and surfaces, which requires the formation of a blending surface. The simplest situation arises when the blending surface reduces to a blending curve, the formation of which requires the calculation of multiple roots of a polynomial $p(y)$. Since the coefficients of this polynomial are, in practical problems, corrupted by noise, its roots are, in general, simple. This property does not, however, reflect design intent – a smooth intersection – but if the noise is sufficiently small, then $p(y)$ is near another polynomial $\tilde{p}(y)$ that has one or more multiple roots, which can therefore be used for the design of a blending curve. Structured low rank approximations of the Sylvester resultant matrices of $p^{(k)}(y)$ and $p^{(k+1)}(y)$, where $p^{(k)}(y)$ is the kth derivative of $p(y)$, $k = 0, 1, \ldots,$ allow the polynomial $\tilde{p}(y)$ and its multiple roots to be computed, and this has been considered for power basis polynomials [7,8]. The work in this paper is therefore a necessary requirement for the extension of this polynomial root solver to the Bernstein basis.

There are important differences between the GCD of $\hat{f}(y)$ and $\hat{g}(y)$, and an AGCD of $f(y)$ and $g(y)$. For example, the GCD of two exact polynomials is unique up to a non-zero constant multiplier, but an AGCD of two inexact polynomials is not unique because it can be defined in several different ways and it is a function of the relative error of the coefficients of $f(y)$ and $g(y)$ [17,18]. It cannot, however, be assumed in practical problems that this error is uniformly distributed across the coefficients, and the implications of this property for AGCD computations are considered in [14]. It is therefore assumed in this paper that the upper bound of the relative error of the coefficients of $f(y)$ and $g(y)$ is a uniformly distributed random number that spans two orders of magnitude.

The Sylvester matrix of two Bernstein polynomials is reviewed in Section 2, and the application of the method of SNTLN to the computation of an AGCD of $f(y)$ and $g(y)$ is considered in Section 3. Examples of the application of the method of SNTLN to the computation of the coefficients of an AGCD of degree t are in Section 4, and Section 5 contains a summary of the paper.

The computation of a structured low rank approximation of the Sylvester matrix of two power basis polynomials has been considered by several researchers [19–23]. The computation of this matrix, $S(f,g)$, for the Bernstein polynomials $f(y)$ and $g(y)$ is sufficiently different to merit a separate investigation because, apart from the importance of the Bernstein basis in CAGD, non-trivial numerical issues that do not occur with power basis polynomials must be considered. In particular, $S(f,g)$ is not Tœplitz, unlike its power basis equivalent, and the combinatorial terms in the Bernstein basis imply that the ratio of the