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A Hierarchical Finite Element Monte Carlo Method for Stochastic Two-Scale Elliptic Equations

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Abstract

We consider two-scale elliptic equations whose coefficients are random. In particular, we study two cases: in the first case, the coefficients are obtained from an ergodic dynamical system acting on a probability space, and in the second the case, the coefficients are periodic in the microscale but are random. We suppose that the coefficients also depend on the macroscopic slow variables. While the effective coefficient of the ergodic homogenization problem is deterministic, to approximate it, it is necessary to solve cell equations in a large but finite size "truncated" cube and compute an approximated effective coefficient from the solution of this equation. This approximated effective coefficient is, however, realization dependent; and the deterministic effective coefficient of the homogenization problem can be approximated by taking its expectation. In the periodic random setting, the effective coefficient for each realization are obtained from the solutions of cell equations which are posed in the unit cube, but to compute its average by the Monte Carlo method, we need to consider many uncorrelated realizations to accurately approximate the average. Straightforward employment of finite element approximation and the Monte Carlo method to compute this expectation with the same level of finite element resolution and the same number of Monte Carlo samples at every macroscopic point is prohibitively expensive. We develop a hierarchical finite element Monte Carlo algorithm to approximate the effective coefficients at a dense hierarchical network of macroscopic points. The method requires an optimal level of complexity that is essentially equal to that for computing the effective coefficient at one macroscopic point, and achieves essentially the same accuracy. The levels of accuracy for solving cell problems and for the Monte Carlo sampling are chosen according to the level in the hierarchy that the macroscopic points belong to. Solutions and the effective coefficients at the points where the cell problems are solved with higher accuracy and the effective coefficients are approximated with a larger number of Monte Carlo samples are employed as correctors for the effective coefficient at those points at which the cell problems are solved with lower accuracy and fewer Monte Carlo samples. The method combines the hierarchical finite element method for solving cell problems at a dense network of macroscopic points with the optimal complexity developed in D. L. Brown, Y. Efendiev and V. H. Hoang, *Multiscale Model. Simul.* **11** (2013), with a hierarchical Monte Carlo sampling algorithm that uses different number of samples at different macroscopic points depending on the level in the hierarchy that the macroscopic points belong to. Proof of concept numerical examples confirm the theoretical results.

1 Introduction

In heterogeneous media, the primary challenge in obtaining accurate simulations that are cost effective is the multiscale nature of the media. This multiscale nature often necessitates the use of very fine scale meshes to resolve the characteristics of the heterogeneities. This difficulty is compounded by the fact that often the uncertainties in the material parameters may be large. This multiscale nature and uncertainty in material properties are exhibited in a variety of applications such as porous media, i.e. filtration or batteries devices for industrial applications [11, 25] or

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