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Fractional Laplace transform method in the framework of the CTIT transformation

Nuri Ozalp, Ozlem Ozturk Mizrak *

Ankara University, Faculty of Sciences, Department of Mathematics, 06100 Besevler, Ankara, Turkey

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ABSTRACT

We propose an adapted Laplace transform method that gives the solution of a linear fractional differential equation with constant coefficients in terms of exponential function. After we mention what the utilized transformation, the CTIT transformation, is based on, we explain how it can reduce the problem from fractional form to ordinary form when it is used with Laplace transformation, via some examples for $0 < \alpha < 2$ where α is the order of fractional derivative. Finally, we illustrate the applications of our results.

1. Introduction

Fractional calculus concept, i.e., the idea of differentiation and integration of arbitrary (not necessary integer) order might be unfamiliar, but it has actually been existing since the very beginning of differential calculus and so has a rich history. Although one can see this extensively discussed history in [1–4], in [4], Kilbas et al. concisely summarized not only the definition of this subject, but also how this subject can be of importance to many fields. They stated: 'The subject of fractional calculus has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering'. Indeed it does provide several potentially useful tools for applications in for example fluid flow, viscoelasticity, chemical physics and signal processing.

Among the various definitions found by many mathematicians, using their own notation and approach, that fit the idea of an arbitrary order integral or derivative, one version that has been popularized in the world of fractional calculus is the Riemann–Liouville definition. Even though this definition is historically the first and the theory about it has been established very well by now, some difficulties emerge when trying to model real-world phenomena using this definition [5]. Therefore, a modified concept of Riemann–Liouville derivative, i.e., Caputo derivative is defined. The essence of Caputo's contribution was also well stated by Connolly. He said that 'Caputo defined a version of the fractional derivative similar to Riemann–Liouville type, in which he changed the order of integration and differentiation enabling the user to apply non-zero initial conditions to the fractional integral with real life measurable quantities' [6]. The Caputo fractional derivative is particularly useful where initial conditions are concerned. More specifically, this fractional derivative enables the initial conditions to be conventionally formulated, so in this study, we consider Caputo type fractional differential equations of order $\alpha > 0$, which seem to be better suited to modeling scientific facts.

For one might approach the finding a solution of a linear fractional differential equation with constant coefficients, two arguments are presented: one involving a direct approach, and one based on the Laplace transform. To the best of

* Corresponding author.
E-mail addresses: nozalp@science.ankara.edu.tr (N. Ozalp), oomizrak@ankara.edu.tr (O.O. Mizrak).

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our knowledge, for the second option, i.e. for the fractional Laplace transform method, solutions of the linear differential equation generally have been constructed in terms of one or two parameter Mittag-Leffler functions, generalized Wright functions or F-functions which are all infinite series and related with each other closely [2,7–11]. On the other hand, from the perspective of engineers or applied scientists, fast and easy-computable techniques are the most preferred choices. Whereas, doing work with infinite series is not as easy as expected, and the question is: Except for rotating around the serial solutions, is there any way that leads us to reduce the solution finding problem for a linear fractional differential equation to a problem in ordinary domain that gives the solution in terms of exponential function rather than Mittag-Leffler function or other infinite series?The main objective of this paper is to test if such an alternative approach is possible rather than implementing Laplace transform method directly and to provide estimates for accuracy of the computations with comparative graphs showing the behavior of the solutions. Indeed the idea of converting fractional differential equations into ordinary differential equations that makes easier using fractional calculus exists in the literature to find exact solutions of the given equation. For example, in [12–15], the author first reduces the solution finding process into a simple form using fractional complex transform and then applies the basic (G'/G)-expansion method for exact solutions of some types of fractional differential–difference equations (DDEs) and for a fractional reaction diffusion Brusselator model. Similarly, in [16], the author combines fractional complex transform with discrete tanh method for exact solutions of a fractional DDE model associated with a nonlinear electrical transmission line.

The article is organized as follows. In Section 2, some preliminaries about fractional calculus are presented. In Section 3, for the convenience of the reader, we give a brief background on the CTIT transformation. Some main results are shown in Section 4. Section 5 is devoted to the examples to show the adapted Laplace transform method works properly making comparisons with the numerical solutions in [4], and a fractional chemical model is introduced as an example. Finally, some conclusions are given in Section 6.

2. Preliminaries

Definition 1. Caputo fractional derivative with order α for a function $x(t)$ is defined as

$$D^{\alpha}_{t_0}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x^{(m)}(\tau)d\tau \tag{2.1}$$

where $0 \le m-1 \le \alpha < m,\ m \in Z^+$. Here and elsewhere $t = t_0$ is the initial time and Γ denotes the Gamma function.

Definition 2. Riemann–Liouville fractional integral of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$I^{\alpha}_{t_0}f(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau. \tag{2.2}$$

Definition 3. The Mittag-Leffler function is defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ka+1)} \tag{2.3}$$

where $\alpha > 0,\ z \in C$. The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ka+\beta)} \tag{2.4}$$

where $\alpha > 0,\ \beta > 0,\ z \in C$.

Definition 4. The general Wright function $p\Psi_q(z)$ [17,18] is defined for $z \in \mathbb{C}$, complex $a_i, b_j \in \mathbb{C}$, and real $\alpha_i, \beta_j \in \mathbb{R}(i = 1, \ldots, p; j = 1, \ldots, q)$ by the series

$$p\Psi_q(z) = p\Psi_q[^{(a_i,\alpha_i)_{1p}}_{(b_j,\beta_j)_{1,q}}, |z] := \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\cdot\frac{z^k}{k!}, \tag{2.5}$$

where $z, a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j \in \mathbb{R}, i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$.

3. Background on the CTIT transformation

It is generally known that as conventional calculus includes just integer order differential and integral operators, it significantly simplify its use for solving applied problems in various fields of science. However, in case of fractional calculus,