Journal of Computational and Applied Mathematics 317 (2017) 218–234

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The minimal angle condition for quadrilateral finite elements of arbitrary degree

Gabriel Acosta [a,*], Gabriel Monzón [b]

[a] Universidad de Buenos Aires, IMAS-CONICET, Departamento de Matematica, Pabellón I Facultad de Ciencias Exactas y Naturales, (1428) Buenos Aires, Argentina

[b] Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutiérrez 1150, (1613) Los Polvorines, Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 17 July 2015

MSC:
65N15
65N30

Keywords:
Quadrilateral elements
Anisotropic finite elements
Lagrange interpolation
Minimum angle condition
Maximum angle condition

ABSTRACT

We study $W^{1,p}$ Lagrange interpolation error estimates for general *quadrilateral* $\mathcal{Q}_k$ finite elements with $k \geq 2$. For the most standard case of $p = 2$ it turns out that the constant C involved in the error estimate can be bounded in terms of the *minimal interior angle* of the quadrilateral. Moreover, the same holds for any p in the range $1 \leq p < 3$. On the other hand, for $3 \leq p$ we show that C also depends on the *maximal interior angle*. We provide some counterexamples showing that our results are sharp.

1. Introduction

This paper deals with error estimates in the $W^{1,p}$ norm for the $\mathcal{Q}_k$ Lagrange interpolation on a general convex quadrilateral $K \subset \mathbb{R}^2$. Denoting the interpolant with Q_k the standard error estimate is usually found in the form

$$\|u - Q_k u\|_{0,p,K} + h\,|u - Q_k u|_{1,p,K} \leq Ch^{k+1}|u|_{k+1,p,K}, \tag{1.1}$$

being h the diameter of K. Inequality (1.1) involves the L^p error estimate

$$\|u - Q_k u\|_{0,p,K} \leq Ch^{k+1}|u|_{k+1,p,K}, \tag{1.2}$$

and the seminorm estimate

$$|u - Q_k u|_{1,p,K} \leq Ch^{k}|u|_{k+1,p,K}. \tag{1.3}$$

A central matter of (1.1) concerns the dependence of C on basic geometric quantities of the underlying element K. It is known that the constant C in (1.2) remains uniformly bounded for arbitrary convex quadrilaterals (see Theorem 6.1). However this statement is false for the constant C in (1.3) (see for instance the counterexamples in the last section). The primary goal of this paper is to study the dependence of C in (1.3) on the *interior angles* of K. Although the role of the interior angles has been related to C in many previous works, none of them, to the best of the authors knowledge, have given a result as plain as the one offered in this paper. For instance, bounding the minimal and the maximal inner angle is considered a central

* Corresponding author.
E-mail addresses: gacosta@dm.uba.ar (G. Acosta), gmonzon@ungs.edu.ar (G. Monzón).

http://dx.doi.org/10.1016/j.cam.2016.11.041
0377-0427/

matter in mesh generation algorithms since the early work by Ciarlet and Raviart [1], however no proof of sufficiency has been given so far (at least for an arbitrary degree of interpolation).

In order to present our results let us first introduce the following classical definition that we write for both triangles and quadrilaterals for further convenience.

Definition 1.1. Let K (resp. T) be a convex quadrilateral (resp. a triangle). We say that K (resp. T) satisfies the *minimum angle condition* with constant $\psi_m \in \mathbb{R}$, or shortly $mac(\psi_m)$, if for any internal angle θ of K (resp. T) $0 < \psi_m \le \theta$.

Our first result says that the constant in (1.3), for a fixed degree k and a fixed value of p with $1 \le p < 3$, can be written as $C = C(\psi_m)$. As a consequence, the same can be stated about the constant in (1.1). This seems to be the most general result available for quadrilaterals in the case $k \ge 2$. In spite of the fact that much weaker geometrical conditions are known to be sufficient for the case $k = 1$, we show, by means of a counterexample, that they fail for a higher degree interpolation. This counterexample also warns that removing the minimal angle condition may indeed lead to a blowing-up constant in (1.3).

The *mac* is the most standard condition considered in textbooks for *triangular* finite elements. Actually, in that case, it is equivalent to the so called *regularity condition*, i.e. equivalent to the existence of a constant σ such that

$$h/\rho \le \sigma, \tag{1.4}$$

where ρ denotes the diameter of the maximum circle contained in T. On the other hand, the term *anisotropic* or *narrow* is usually applied to elements that do not satisfy (1.4). Even when *triangles* can become narrow only if the minimal angle is approaching zero a very different situation occurs on quadrilaterals. Indeed, in that case the *mac* condition is less restrictive than (1.4) since arbitrarily narrow elements are allowed with a positive uniform bound for the minimal angle (for example, anisotropic rectangles always verify $mac(\pi/2)$). Anisotropic elements are important for instance in problems involving singular layers and the first works dealing with them arise during the seventies showing that (1.4) can be replaced (for triangles) by the weaker following condition.

Definition 1.2. Let K (resp. T) be a convex quadrilateral (resp. triangle), we say that K (resp. T) satisfies the *maximum angle condition* with constant $\psi_M \in \mathbb{R}$, or shortly $MAC(\psi_M)$, if for any internal angle θ of K (resp. T) $\theta \le \psi_M < \pi$.

Indeed, in [2,3] it is proved that the *MAC* is sufficient to have optimal order error estimates for Lagrange interpolation on *triangles*. In the case of quadrilateral elements, (1.4) it is also a sufficient condition as it was shown by Jamet [4] for $k = 1$ and $p = 2$. This condition is less restrictive than that proposed in [1] where the authors require the existence of two positive constants μ_1, μ_2 such that

$$h/s \le \mu_1 \tag{1.5}$$

where s is the length of the shortest side of K, and

$$|\cos(\theta)| \le \mu_2 < 1 \tag{1.6}$$

for each inner angle θ of K. Observe that under the regularity condition (1.4) the quadrilateral can degenerate into a triangle (for instance if the shortest side tends to zero faster than their neighboring sides or if the maximum angle of the element approaches π), however this kind of quadrilateral cannot become too narrow. Condition (1.6) will play an important role in the sequel and therefore we introduce the following alternative definition.

Definition 1.3. We say that a quadrilateral K satisfies the double angle condition with constants ψ_m, ψ_M, or shortly $DAC(\psi_m, \psi_M)$, if K verifies $mac(\psi_m)$ and $MAC(\psi_M)$ simultaneously, i.e., if all inner angles θ of K verify $0 < \psi_m \le \theta \le \psi_M < \pi$.

The *DAC* allows anisotropic quadrilaterals (such as narrow rectangles) as well as families of quadrilaterals that may degenerate into triangles. To see that consider, for instance, a quadrilateral with vertices $(0, 0)$, $(1, 0)$, $(s, 1-s)$ and $(0, 1-s)$ and take $0 < s \to 0$.

For anisotropic quadrilaterals several papers have been written mainly in the *isoparametric* case with $k = 1$. In [5,6] narrow quadrilaterals are studied and the authors require the two longest sides of the element to be opposite and almost parallel, the constant C obtained by them depends on an angle which *in some cases* is the minimum angle of the element. *Anisotropic error estimates* for some perturbations of rectangles have been derived in [7,8]. On the other hand, for $k = 1$, more general and subtle conditions can be found in the literature. For $k = 1$ and $p = 2$, it is proved in [9] that the optimal error estimate (1.3) can be obtained under the following weak condition.

Definition 1.4. Let K be a convex quadrilateral, and let d_1 and d_2 be the diagonals of K. We say that K satisfies the *regular decomposition property* with constants $N \in \mathbb{R}$ and $0 < \psi_M < \pi$, or shortly $RDP(N, \psi_M)$, if we can divide K into two triangles along one of its diagonals, that will be called always d_1, in such a way that $|d_2|/|d_1| \le N$ and both triangles have its maximum angle bounded by ψ_M.

In Remarks 2.3–2.7 of [9] it is shown that the regular decomposition property *RDP* is certainly much weaker than those considered in previous works (including [7,8,1,4–6]). We collect for further reference some elementary remarks.