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Least-squares collocation for linear higher-index differential–algebraic equations

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ABSTRACT

Differential–algebraic equations with higher index give rise to essentially ill-posed problems. Therefore, their numerical approximation requires special care. In the present paper, we state the notion of ill-posedness for linear differential–algebraic equations more precisely. Based on this property, we construct a regularization procedure using a least-squares collocation approach by discretizing the pre-image space. Numerical experiments show that the resulting method has excellent convergence properties and is not much more computationally expensive than standard collocation methods used in the numerical solution of ordinary differential equations or index-1 differential–algebraic equations. Convergence is shown for a limited class of linear higher-index differential–algebraic equations.

1. Introduction

In the present paper, we consider boundary value problems (BVPs) for linear differential–algebraic equations (DAEs)

$$A(t)(Dx)'(t) + B(t)x(t) = q(t), \quad t \in [a, b], \tag{1}$$

$$G_a x(a) + G_b x(b) = r. \tag{2}$$

Here, $[a, b] \subset \mathbb{R}$ denotes a finite interval, $q : [a, b] \to \mathbb{R}^m$ is a sufficiently smooth vector-valued function, $B : [a, b] \to \mathbb{R}^{m\times m}$, $A : [a, b] \to \mathbb{R}^{m\times k}$ are at least continuous but sufficiently smooth matrix-valued functions, and $D : [a, b] \to \mathbb{R}^{k\times m}$ is at least continuously differentiable. Moreover, $G_a, G_b \in \mathbb{R}^{l\times m}$ and $r \in \mathbb{R}^l$. Thereby, l is the dynamical degree of freedom of the DAE, that is, the number of free parameters of the general solution of the DAE (e.g., [1, Section 2], [2, Section 2.6]), which can be fixed by boundary conditions. Initial value problems (IVPs) are incorporated by $G_b = 0$. We suppose $0 \le l \le k < m$. If $l = 0$ then there are no free parameters and no boundary condition will be given.

We are interested in solutions $x : [a, b] \to \mathbb{R}^m$ satisfying the BVP (1)–(2) in a sense which will be specified in Section 2.

Here, for clarity of the presentation, we focus on DAEs featuring variables partitioned into differentiated and algebraic components by assuming a constant matrix function D and the leading term of the special form,

$$D = [\,I \;\; 0\,], \qquad \operatorname{rank} D = k, \qquad \operatorname{rank} A(t) = k, \quad t \in [a, b]. \tag{3}$$

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In particular, this is the case for all semi-explicit DAEs. The first k components of the unknown function x are the *differentiated* components and the subsequent $m-k$ components are traditionally called the *algebraic* components. We emphasize that no derivatives of the algebraic components appear in the DAE. We refer to Section 5.1 for more general DAEs.

Collocation methods using piecewise polynomial ansatz functions are well-established and robust numerical methods to approximate BVPs in explicit ordinary differential equations and index-1 DAEs, which are well-posed in their natural Banach spaces, see [3,1] for the respective comprehensive surveys.

There are different possibilities on how to make the collocation ansatz, see [1, Section 3]. Below we describe only one of these possibilities which we address later on and which is, for instance, implemented in COLDAE [4]. Basically, as an ansatz for the differentiated components, we use continuous piecewise polynomial functions of a certain degree and for the algebraic components, generally discontinuous piecewise polynomial functions, whose degree is lower by one.

Let $n \in \mathbb{N}$ and consider the following partition of the interval $[a, b]$:

$$a = t_0 < t_1 < \cdots < t_n = b.$$

For $K \geq 0$, let us denote by $\mathscr{P}_K$ the set of all polynomials of degree less than or equal to K.

We fix a certain integer $N \geq 1$ and approximate the differentiated solution components $x_1, \ldots, x_k$ by continuous, piecewise polynomial functions of degree N with possible breakpoints at $t_1, \ldots, t_{n-1}$, while we approximate the algebraic components $x_{k+1}, \ldots, x_m$ by possibly discontinuous piecewise polynomial functions of degree $N-1$ with possible jumps at $t_1, \ldots, t_{n-1}$. Consequently, we search for the numerical approximation p in the function set X_n,

$$\begin{aligned} X_n = \{p \in L^2(a,b)^m :\ & p_\kappa \in C[a,b],\ p_\kappa|_{[t_{j-1},t_j)} \in \mathscr{P}_N,\ \kappa = 1,\ldots,k, j = 1,\ldots,n, \\ & p_\kappa|_{[t_{j-1},t_j)} \in \mathscr{P}_{N-1},\ \kappa = k+1,\ldots,m, j = 1,\ldots,n\}. \end{aligned} \tag{4}$$

By construction, $p \in X_n$ implies $Dp \in C[a, b]^k$. Since X_n has dimension $Nmn+k$, $Nmn+k$ conditions are necessary to uniquely determine $p \in X_n$.

The collocation points t_{ji} are specified by choosing N values

$$0 < \rho_1 < \cdots < \rho_N < 1,$$

and setting $t_{ji} := t_{j-1} + \rho_i h_j,\ j = 1, \ldots, n,\ i = 1, \ldots, N$, where $h_j = t_j - t_{j-1}$. This choice prevents redundant equations (cf., [1, Section 3]), but excludes Lobatto and Radau collocation points. Note that in COLDAE, Gaussian points are used [4].

In order to determine the discrete solution $p \in X_n$, the classical collocation method is applied directly to the DAE system,

$$A(t_{ji})(Dp)'(t_{ji}) + B(t_{ji})p(t_{ji}) = q(t_{ji}), \quad i = 1, \ldots, N, j = 1, \ldots, n \tag{5}$$

$$G_a p(a) + G_b p(b) = r. \tag{6}$$

It follows immediately that (5)–(6) consists of $Nmn + l$ conditions to determine p. For index-1 DAEs, we have $l = k$ and therefore, the system (5)–(6) is well-balanced in the sense that the number of equations equals the number of unknowns. In case of higher-index DAEs, l is less than k and for a balanced system, extra boundary conditions have to be prescribed.[1]

In this context, IVPs and BVPs are treated in the same way, more precisely, IVPs are treated as BVPs. Here the substance inheres in the DAE and it is a secondary matter whether we have initial conditions or boundary conditions.

This kind of collocation directly applied to the given BVP works well for index-1 DAEs and also for a limited class of index-2 DAEs via *projected collocation* [4,1]. Here, we are interested in a direct treatment of general higher-index DAEs, without any preliminary and incorporated index reduction procedures as applied, for instance, in [5,6].

We now report on experiments with direct collocation for two simple academic examples of index 2 and index 3. Both examples are known to cause serious difficulties in the numerical integration depending on the involved parameters. All computations have been carried out in MATLAB.[2] The ansatz polynomials for the κth component of p on the subinterval $[t_{j-1}, t_j)$ have been represented as

$$p_\kappa|_{[t_{j-1},t_j)}(t) = z_{n,\kappa,0} + h\sum_{s=1}^{N} z_{n,\kappa,s}\bar{\psi}_s\left(\frac{t-t_{j-1}}{h}\right), \quad \kappa = 1,\ldots,k,$$

$$p_\kappa|_{[t_{j-1},t_j)}(t) = \sum_{s=1}^{N} z_{n,\kappa,s}\psi_s\left(\frac{t-t_{j-1}}{h}\right), \quad \kappa = k+1,\ldots,m,$$

1 In particular, for DAEs in Weierstraß–Kronecker form, one has $k = l + \sum_{i=1}^{s}(l_i - 1)$, where s is the number of Jordan blocks in the nilpotent part and l_i is their size, cf., Section 3.1. Obviously, if there exists a $l_{i_*} \geq 2$, which means that the DAE has Kronecker index $\mu := \max\{l_i : i = 1, \ldots, s\} \geq 2$, then it results that $k > l$.

In general we apply here the tractability index which generalizes the Kronecker index, e.g., [2], see also Definition 2.1. We refer to [2] for relations to other index notions such as differentiation index and strangeness index.

2 MATLAB Release 2013b, The MathWorks, Inc., Natick, Massachusetts, United States.