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A hierarchical scale separation approach for the hybridized discontinuous Galerkin method

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ABSTRACT

In this work, the hierarchical scale separation (HSS) method developed for linear systems resulting from discontinuous Galerkin (DG) discretizations has been extended to hybridized discontinuous Galerkin (HDG) schemes. The HSS method is related to *p*-multigrid techniques for DG systems but is conceptually much simpler. Our extension of the HSS scheme to the HDG method tested using a convection–diffusion equation for a range of benchmark problems demonstrated excellent performance on a par with that of the HSS method for a non-hybridized DG approximation. In the limiting case of a pure convection equation, the measured convergence rate of the HSS scheme was significantly better than the rates demonstrated in the non-hybridized case.

1. Introduction

Discontinuous Galerkin (DG) methods [1] combine the most attractive features of the finite volume (local conservation, robustness for advection-dominated problems, shock-capturing features, etc.) and the finite element (high order discretizations, Galerkin formulation, etc.) methods. However, the price is a computationally more expensive scheme, partly due to a significantly greater number of degrees of freedom than in a finite element discretization of equal order, partly caused by more expensive evaluations of element and boundary integrals. This performance disadvantage has even more impact if a linear equation system resulting from a DG discretization must be solved as a part of the solution algorithm—e.g., in an implicit time-stepping scheme.

A number of recent publications (see, e.g., [2–8]) introduced the hybridizable discontinuous Galerkin (HDG) methods aiming to address this drawback. The main idea of the HDG method lies in the introduction of an approximation space for traces of primary unknowns on element boundaries relying on classical ideas in the context of hybrid mixed methods, see, e.g., [9,10]. This approximation space on the mesh 'skeleton' constitutes a globally connected problem, whereas the original system unknowns are computed in a postprocessing step by solving element-local problems that utilize those traces. This approach substantially reduces the number of global degrees of freedom for higher order discretizations, lends itself readily to an efficient parallelization, and even sometimes results in better convergence rates than its non-hybridizable counterparts [2].

In order to speed up the linear system solves for DG discretizations, a hierarchical scale separation (HSS) approach has been introduced in [11] for a nonsymmetric interior penalty Galerkin (NIPG) method. The main idea of the HSS technique is related to the *p*-multigrid scheme proposed for DG in [12] that, in its turn, was motivated by a similar approach introduced

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for the spectral method in [13]. Contrary to the classical multigrid method that relies on a mesh hierarchy to suppress the low wave number errors present in the finest-mesh solution, the *p*-multigrid method uses in the same role a hierarchy of approximation spaces on a fixed mesh (usually corresponding to DG spaces of different polynomial order). The hierarchical scale separation (HSS) method goes even further and resorts to a two-space technique resulting in a much simpler algorithm and a more complete separation of fine- and coarse-scale solutions than the *p*-multigrid method. The coarse-scale problem has the computational structure of a cell-centered finite volume method and is solved globally. The fine-scale problems are computed as local corrections to the coarse-scale solution; thus producing a numerical algorithm highly suitable for an efficient parallel implementation. In [11], the HSS method demonstrated performance on a par or exceeding that of the *p*-multigrid method. The main promise of this type of method lies in a much reduced parallel communication overhead as compared to traditional linear solvers; this advantage is expected to become even more significant for massively parallel applications.

In the present paper, we extend the HSS paradigm to an HDG discretization of a convection–diffusion equation and evaluate the performance of the method on a range of benchmark problems of varying complexity. The sensitivity of the method with respect to the diffusion coefficient and the stabilization parameter is investigated for selected test cases. The combination of the two aforementioned numerical techniques both aiming to enhance the computational performance may cover a lot of ground toward our goal: Making DG discretizations more competitive with the classical finite element and finite volume methods for a range of important applications.

The remainder of this paper is organized as follows. In Section 2, we formulate the boundary value problem for a generic convection–diffusion equation and discretize it using an HDG method. Section 3 details the HSS algorithm. Some numerical examples illustrating the performance of the proposed method are presented in Section 4. Conclusions are offered in Section 5.

2. The hybridized discontinuous Galerkin method

In this work, we consider the two-dimensional linear convection–diffusion equation on a bounded domain $\Omega \subset \mathbb{R}^2$,

$$-\varepsilon \Delta u + \nabla \cdot (\mathbf{c}u) = f, \quad \forall \mathbf{x} \in \Omega,\ u = g,\ \forall \mathbf{x} \in \partial\Omega_D \tag{1}$$

for a constant vector $\mathbf{c} \in \mathbb{R}^2$, a constant scalar $\varepsilon \in \mathbb{R}_+$ and functions $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. $\partial\Omega_D$ denotes the Dirichlet boundary which encompasses the whole domain boundary in the convection–diffusion case, i.e. $\partial\Omega_D = \partial\Omega$ if $\varepsilon > 0$, or is limited to the inflow part of the external boundary $\partial\Omega_D = \{\mathbf{x} \in \partial\Omega \mid \mathbf{c} \cdot \mathbf{n} \leq 0\}$ for the pure convection $\varepsilon = 0$, where $\mathbf{n}$ denotes an exterior unit normal to $\partial\Omega$.

As is frequently done for DG-type methods, we rewrite this equation as a first order system

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla u, \\ \nabla \cdot (\mathbf{c}u - \varepsilon\boldsymbol{\sigma}) &= f. \end{aligned}$$

The presentation of the HDG method below follows closely [2]. First, we need to define the standard DG approximation space V_h on a triangulation $\Omega = \bigcup_{k=1}^{\mathtt{ne}} \Omega_k$ as

$$V_h := \{f \in L^2(\Omega) \mid f_{|\Omega_k} \in \Pi^p(\Omega_k),\ k = 1, \ldots, \mathtt{ne}\},$$

where Π^p is the space of polynomials of total degree at most p. Approximating the unknown u on the skeleton of the mesh necessitates the introduction of yet another space, the so-called hybrid ansatz space. So let Γ denote the set of all edges of the mesh, and let $\Gamma = \bigcup_{k=1}^{\mathtt{nf}} \Gamma_k$. (Every edge only occurs once, independent of whether it is a boundary edge or not.) Then, M_h is defined by

$$M_h := \{f \in L^2(\Gamma) \mid f_{|\Gamma_k} \in \Pi^p(\Gamma_k),\ k = 1, \ldots, \mathtt{nf}\}.$$

For a point $\mathbf{x} \in \partial\Omega_k$, we define the one-sided values of a scalar quantity $w = w(\mathbf{x})$ by

$$w^-(\mathbf{x}) := \lim_{\varepsilon \to 0^+} w(\mathbf{x} - \varepsilon\,\mathbf{n}) \quad \text{and} \quad w^+(\mathbf{x}) := \lim_{\varepsilon \to 0^+} w(\mathbf{x} + \varepsilon\,\mathbf{n}),$$

respectively. (Obviously, at the physical boundary of the domain, only the first expression is well defined.) The one-sided values of a vector-valued quantity $\mathbf{v}$ are defined analogously. Then using the standard DG notation, the *average* and the *jump* of w and $\mathbf{v}$ in $\mathbf{x}$ are given by

$$\begin{aligned} \{w\} &:= \frac{w^- + w^+}{2} \quad \text{and} \quad [\![w]\!] := w^-\mathbf{n} - w^+\mathbf{n}, \\ \{\mathbf{v}\} &:= \frac{\mathbf{v}^- + \mathbf{v}^+}{2} \quad \text{and} \quad [\![\mathbf{v}]\!] := \mathbf{v}^- \cdot \mathbf{n} - \mathbf{v}^+ \cdot \mathbf{n}, \end{aligned}$$

where $\mathbf{n}$ denotes an exterior unit normal to Ω_k. Note that $[\![w]\!]$ is a vector-valued quantity, and $[\![\mathbf{v}]\!]$ is a scalar.

The distinct feature of the HDG method is the fact that only the degrees of freedom contained in M_h occur in the globally connected system [2]. In order to clarify this, we introduce the so-called non-homogeneous local solves as follows: Find