ARTICLE IN PRESS

Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Optimal error analysis of Crank–Nicolson schemes for a coupled nonlinear Schrödinger system in 3D

Weiwei Sun [a], Jilu Wang [b,*]

[a] Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong
[b] Department of Scientific Computing, Florida State University, Tallahassee, FL 32306, USA

ARTICLE INFO

Article history:
Received 1 January 2015
Received in revised form 5 October 2016

Keywords:
Unconditionally optimal error estimate
Coupled nonlinear Schrödinger system
Semi-implicit Crank–Nicolson schemes

ABSTRACT

The paper is concerned with the time step condition of the commonly-used semi-implicit Crank–Nicolson finite difference schemes for a coupled nonlinear Schrödinger system in three dimensional space. We present the optimal L^2 error estimate without any restriction on time step, while all previous works require certain time step conditions. Our approach is based on a rigorous analysis in both real and imaginary parts of the energy estimate (inequality) of the error function. Numerical examples for both two-dimensional and three-dimensional models are investigated and numerical results illustrate our theoretical analysis.

1. Introduction

In this paper, we consider the initial–boundary value problem of a coupled nonlinear Schrödinger (CNLS) system in three dimensional space:

$$\mathrm{i}u_t + \Delta u + (|u|^2 + \beta|v|^2)u = 0, \quad x \in \Omega,\ 0 < t \le T, \tag{1.1}$$

$$\mathrm{i}v_t + \Delta v + (|v|^2 + \beta|u|^2)v = 0, \quad x \in \Omega,\ 0 < t \le T, \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \quad x \in \Omega, \tag{1.3}$$

$$u|_{x\in\partial\Omega} = 0, \qquad v|_{x\in\partial\Omega} = 0, \tag{1.4}$$

where $\mathrm{i} = \sqrt{-1}$, β is a given constant, $\Omega = [0, L_1] \times [0, L_2] \times [0, L_3] \subset \mathbb{R}^3$. $u(x, t)$ and $v(x, t)$ are complex unknown functions defined in $\Omega \times [0, T]$. When $\beta = 1$ and $u_0(x) = v_0(x)$, the system reduces to a single equation

$$\mathrm{i}u_t + \Delta u + 2|u|^2 u = 0, \quad x \in \Omega,\ 0 < t \le T, \tag{1.5}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.6}$$

$$u|_{x\in\partial\Omega} = 0. \tag{1.7}$$

The Schrödinger equations may describe many physical phenomena in optics, mechanics, and plasma physics. Here, we are particularly interested in coupled nonlinear Schödinger system due to its important applications and physical significance [1–3]. We refer the readers to [4–6] for an overview of various properties of the system, such as existence, uniqueness

* Corresponding author.
E-mail addresses: maweiw@math.cityu.edu.hk (W. Sun), jwang13@fsu.edu (J. Wang).

http://dx.doi.org/10.1016/j.cam.2016.12.004
0377-0427/

Please cite this article in press as: W. Sun, J. Wang, Optimal error analysis of Crank–Nicolson schemes for a coupled nonlinear Schrödinger system in 3D, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.12.004

and stability results of solutions. Numerical methods and analysis for solving such equations have been investigated extensively, *e.g.*, see [7–14] for finite difference methods, [15,16] for finite element methods and [17–22] for others. Usually, fully implicit schemes are unconditionally stable. However, at each time step, one has to solve a system of nonlinear equations. An explicit scheme is much easy for implementation. But it suffers the severely restricted time stepsize from the convergence requirement. A popular and widely-used approach is a semi-implicit scheme, such as linearized Crank–Nicolson scheme. At each time step, the scheme only requires the solution of a linear system. The time step restrictive condition is always a key issue for these schemes. For the Schrödinger equations, several modified explicit schemes were studied in [23,24], in which an extra dissipative term was added in the scheme to improve their stability condition. Analysis for implicitly nonlinear schemes can be found in [25,26]. Chang et al. [27] presented a systematic review and numerical comparison of several commonly-used finite difference schemes for the generalized nonlinear Schrödinger equations. Based on their numerical results and comparison, the linearized Crank–Nicolson scheme showed the best performance. Numerical analysis for the linearized Crank–Nicolson scheme was studied by several authors. Wang et al. [28] analyzed this linearized Crank–Nicolson scheme for the CNLS equations in one-dimensional space and provided optimal L^2 error estimate under the time-step restrictive condition $\tau = o(h^{1/4})$, where τ and h are the stepsizes in the temporal direction and the spatial direction, respectively. Applying their approach to the equations in three-dimensional space will require a stronger time-step condition $\tau = o(h^{3/4})$. A similar work for the Kuramoto–Tsuzuki equation can be found in [29] with the same time-step condition. The approach is based on the analysis of the imaginary part of the classical energy estimate (inequality) since the real part is too weak. Such an approach was used also for Galerkin finite element methods, *e.g.*, see [15], where optimal error estimates were obtained under some similar time-step conditions. More recently, Bao and Cai [30] studied a class of Schrödinger equations with an extra ϵ-perturbed term, which reduces to the classical nonlinear Schrödinger equations when $\epsilon = 0$. An optimal L^2 error estimate of a linearized Leap-frog scheme for the equations in one-dimensional space was obtained when $\tau, h \le s_0$ for certain small positive constant s_0. As they pointed out, their analysis can be extended to the equations in three-dimensional space under the time-step condition $\tau = o(h)$ and the results are still valid for the classical Schrödinger equations ($\epsilon = 0$). However, in the analysis the numerical solution in L^∞ norm was bounded, in terms of the classical inverse inequality

$$\|e_h\|_{L^\infty} \le C\gamma_d(h)\|e_h\|_{H^1}$$

for the error function e_h, where d is the dimension and $\gamma_2 = |\ln h|$, $\gamma_3 = h^{-1/2}$. The H^1 error bound $\|e_h\|_{H^1}$ was estimated by using an H^2-estimate method, which requires higher regularity of solution. Also the H^2-estimate method may not be applicable directly to Galerkin finite element methods. To obtain an L^2 error estimate directly by following the classical energy estimate method [28,30], one has to use the inverse inequality

$$\|e_h\|_{L^\infty} \le Ch^{-d/2}\|e_h\|_{L^2}$$

to bound the numerical solution in L^∞ norm, which results in a stronger time step restriction. In practical computations, time-step restrictive conditions may result in the use of an unnecessarily small time step and extremely time-consuming. Also the problem becomes more serious when a non-uniform mesh is used.

The paper focuses on unconditionally optimal error analysis of two popular linearized semi-implicit Crank–Nicolson finite difference schemes for the CNLS system in three dimensional space. In these two schemes, the nonlinear term is treated by a linearized semi-implicit approximation and an explicit approximation, respectively. The optimal L^2 error estimate is obtained without any time-step condition, i.e., $0 < h_r \le L_r$ and $0 < \tau \le T$, for the first scheme and with the condition $h < s_0$ for some small constant $s_0 > 0$ for the second scheme. The approach is based on a rigorous analysis in both real and imaginary parts of the energy estimate (inequality) and a simple inequality, with which the error function at a given time level is bounded, in terms of its average at two consecutive time levels, when $\tau \ge h$. Numerical results presented in this paper confirm that the first scheme is unconditionally convergent. More important is that our approach can be easily extended to Leap-frog finite difference scheme and Galerkin finite element methods to obtain optimal error estimates without any time-step conditions, while those previous works always require certain time-step restrictions.

The paper is organized as follows. In Section 2, we present two linearized Crank–Nicolson schemes and our main results. By introducing some notations and lemmas, we prove optimal error estimates of the finite difference schemes in Section 3. Numerical results are given in Section 4. Two artificial examples in 2D and 3D spaces are presented, respectively, to show that the linearized Crank–Nicolson scheme provides second-order accuracy in both time and spatial directions without any time step conditions. An example for the interaction of two solitons is also presented.

2. Linearized Crank–Nicolson schemes and main results

In this section, we present a linearized Crank–Nicolson scheme. Let $\Omega_h = \{(x_{1,j_1}, x_{2,j_2}, x_{3,j_3}) | x_{1,j_1} = j_1h_1, x_{2,j_2} = j_2h_2, x_{3,j_3} = j_3h_3; 0 \le j_r \le M_r, r = 1, 2, 3\}$ be a partition of Ω with the mesh size $h_r = L_r/M_r$, and $\alpha_1 \le \frac{h_i}{h_j} \le \alpha_2$, $1 \le i, j \le 3$, for some positive constants α_1 and α_2. We denote $h = \max\{h_1, h_2, h_3\}$. Let $\Omega_\tau = \{t_n | t_n = n\tau; 0 \le n \le N\}$ be a uniform partition of $[0, T]$ with the time step $\tau = T/N$, $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$ and denote

$$J_h = \{(j_1, j_2, j_3) | 0 \le j_r \le M_r; r = 1, 2, 3\}$$
$$J_h' = \{(j_1, j_2, j_3) | 1 \le j_r \le M_r - 1; r = 1, 2, 3\}$$
$$J_h'' = \{(j_1, j_2, j_3) | 0 \le j_r \le M_r - 1; r = 1, 2, 3\}.$$

Please cite this article in press as: W. Sun, J. Wang, Optimal error analysis of Crank–Nicolson schemes for a coupled nonlinear Schrödinger system in 3D, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.12.004