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Evaluation of finite part integrals using a regularization technique that decreases instability

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ABSTRACT

A hypersingular integral can be regularized by replacing the whole integrand by a forward difference quotient of 2nd order. If the density function is nearly singular, then Gauss quadrature formulas associated with a suitable modification of the Chebyshev weight function allow to obtain great precision with few nodes. However, in most cases, the own nature of this procedure makes unpredictable the location of quadrature nodes. This paper presents a simple but effective technique whose aim is to mitigate instability when some node lies too close to the pole. Some numerical examples are shown to evaluate the performance of the proposed method.

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1. Introduction

A hypersingular integral over the interval $(-1, 1)$ is defined as the following limit (if exists).

$$\oint_{-1}^{1} \frac{F(t)dt}{(t-x)^2} = \lim_{\varepsilon \to 0} \left(\int_{-1}^{x-\varepsilon} \frac{F(t)dt}{(t-x)^2} + \int_{x+\varepsilon}^{1} \frac{F(t)dt}{(t-x)^2} - \frac{2F(x)}{\varepsilon} \right), \tag{1}$$

where $x \in (-1, 1)$ is the Hadamard-type singularity of order two.

To ensure the existence of (1), it suffices that $F' \in Lip_\alpha$, $0 < \alpha \leq 1$, although this assumption can be weakened [1].

The calculation of singular boundary integrals is a problem that arises from the application of the boundary element method (BEM), and is related to the evaluation of (1). Indeed, a non-trivial discretization process must be carried out to transform a boundary integral, defined on a two-dimensional region, into one of type (1) (cf. [2,3]). On the role played by these integrals in elasticity, mechanics, etc., we refer the reader to [4,5], and references therein.

The following approximation formula is equivalent to (1) (cf. [6]).

$$\oint_{-1}^{1} \frac{F(t)dt}{(t-x)^2} = \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{1} \frac{(t-x)^2 F(t)dt}{((t-x)^2+\varepsilon^2)^2} - \frac{\pi F(x)}{2\varepsilon} \right) - \frac{F(x)}{1-x^2}. \tag{2}$$

Note that both (1) and (2) are approximation formulas depending on a parameter ε. They have a theoretical value but are not appropriate for numerical calculation. One fact is that a great variety of methods to evaluate integrals with strong singularities are currently known, most of which have been published after the sixth decade of the 20th century (cf. [5,7,8]).

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When ε is small, the integrals on the right side of (2) are nearly hypersingular, an issue also treated by several authors due to the important role they play in the applications of the BEM (cf. [9,10]).

Our approach is mainly based on replacing the integrand of the integral defined in (1) by the following forward difference quotient.

$$E(x,t) = \frac{F(t) - F(x) - F'(x)(t-x)}{(t-x)^2}. \tag{3}$$

Paget [11] seems to be one of the first in using (3) to remove the singularity of (1). Nevertheless, when using finite precision arithmetic and $x \approx t$, then $E(x,t)$ is unstable, even when $F''(x)$ exists.

Other than (3), there are diverse regularization techniques that can be applied to (1). Following the same approach as that indicated by (3), if the order of the Hadamard-type singularity is $m > 2$, then one may use the Taylor polynomial of $F(t)$ with degree $m-1$, and centered at $t = x$ (cf. [12]). However, a more effective variant seems to be the use of the polynomial $P(\rho)$ that interpolates the non-singular part of the integral at a uniform mesh of points, and ρ is the distance between the singularity and the integration variable (cf. [2,3]). According to the results reported in [2,3], a maximum accuracy of 8 decimal digits can be achieved when using a sixth-order Gauss quadrature formula.

Regardless of the robustness shown by the method used in [2], the quality of the results should be further enhanced using the barycentric formulation based on n-point sets with an asymptotic density proportional to the Chebyshev weight function of the first kind as $n \to \infty$. A well known fact is that polynomial interpolation based on equidistant points is ill-conditioned [13].

All the above mentioned methods assume that $F(t)$ is regular, and focus on removing the polar singularity. Instead, we are interested in evaluating (1) when F behaves poorly, a problem whose solution is connected with the numerical stability of (3). Then, our starting point is to assume that $F(t) = g(t)H(t)$, where H represents the component of F that is related to numerical instability. In short, H is nearly-singular and g is smooth.

In a previous paper [14] we used Gauss quadrature formulas associated with a suitable modification of the Chebyshev weight function to evaluate the integral of the parametric function (3). The calculation of the corresponding quadrature weights and nodes relies on the existing relation between the modified moments and the coefficients of the Chebyshev series expansion of $H(t)\sqrt{1-t^2}$. As a result, the nodes are not known in advance, since they depend on the function $H(t)$ we have selected. The only information available a priori is that the asymptotical distribution of nodes is $1/(\pi\sqrt{1-t^2})$ [15]. This may cause that some node is located very close to the parameter x, with the consequent loss of digits (cf. [5,16,17]).

In line with previous comments, the purpose of this work is to show how the forward difference quotient (3) can be integrated efficiently regardless of the location of quadrature nodes and singular points. For this we propose a new type of regularization of the integral (1) which consists in conveniently introducing an additional parameter within (3). All of which is organized in the paper as follows.

As for the problem of calculating the parameters of the quadrature formula, we describe in Section 2 a fairly general method that improves the results obtained in [14]. This section also addresses some aspects of Gauss quadrature formulas associated with a weight function partially modified by a rational function, a topic suggested by W. Gautschi in 2004. In Section 3 we describe the ε-regularization that we apply when the distance between nodal and collocation points is very small. Some examples are given in Section 4 to show the performance of our method. Finally, Section 5 contains some remarks as conclusion.

2. Preliminaries

Let us put $W_\pi(t) = W(t)/\pi$, where $W(t) = 1/\sqrt{1-t^2}$.

Let f be a real function defined on $[-1, 1]$. If f is bounded, then the uniform norm of f is $\|f\| = \sup\{|f(t)|;\ t \in [-1,1]\}$.

Let $\|f\|_{p,W} = \left(\int_{-1}^{1} |f(x)|^p W(x)dx\right)^{1/p}$, where $p \geq 1$. Let $L_{p,W}$ denote the vector space of measurable functions f such that $\|f\|_{p,W} < \infty$.

The symbol T_n stands for the nth orthogonal polynomial associated with W, defined as $T_n(x) = \cos(n\theta)$, where $x = \cos(\theta)$.

2.1. Notes on Chebyshev series

Let $\sum_{j=0}^{\prime\infty} c_j T_j(x)$ be the Chebyshev series expansion of $f \in L_{2,W}$, where the prime indicates that the first term in the sum is halved. Moreover, the coefficients c_j are given by (cf. [18, section 5.2])

$$c_j = 2\int_{-1}^{1} T_j(t) f(t) W_\pi(t)dt. \tag{4}$$

The following lemma collects three different results which link the speed of convergence of $S_n(f)$ (the nth partial sum of the Chebyshev series expansion of f) with the smoothness of f (cf. [18, Theorems 5.2, 5.14, 5.16]).

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