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# On the construction of Lyapunov functions with computer assistance

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## HIGHLIGHTS

- A systematic procedure of Lyapunov functions around fixed points of dynamical systems with computer assistance.
- Validation of Lyapunov functions and their domains as "mathematically rigorous" objects.
- A topological property of Lyapunov domains providing a re-parameterization of trajectories, which ensures to validate blow-up solutions (e.g., Takayasu et al. (2017)).
- Numerical validation results exhibit the applicability of our method both for continuous and discrete dynamical systems.

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## ABSTRACT

This paper aims at applications of Lyapunov functions as tools for analyzing concrete dynamical systems with computer assistance, even for non-gradient-like systems. We want to know concrete form of Lyapunov functions around invariant sets and their domains of definition for applying Lyapunov functions to various analysis of both continuous and discrete dynamical systems. Although there are several abstract results for the existence of Lyapunov functions, they cannot induce a systematic and concrete procedure of Lyapunov functions with explicit forms. In this paper, we present a numerical verification method which can validate Lyapunov functions with explicit forms and their explicit domains of definition, which can be applied to arbitrary dynamical systems with (hyperbolic) equilibria or fixed points. The proposed procedure provides us with a powerful validation tool for analyzing asymptotic behavior of dynamical systems.


## 1. Introduction

In this paper, we provide a systematic method to validate the domain of Lyapunov functions around hyperbolic fixed points both for continuous and for discrete dynamical systems with computer assistance, which aims at demonstrating the applicability of Lyapunov functions as a tool for studying dynamical systems.

First of all, we recall the definition of Lyapunov functions for flows in the typical sense. Let $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a flow on $\mathbb{R}^n$.

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**Definition 1.1** (*E.g. [1]*)**.** Let $U \subset \mathbb{R}^n$ be an open subset. Consider the differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad f \in C^1(\mathbb{R}^n, \mathbb{R}^n). \tag{1.1}$$

A *Lyapunov function* $L : U \to \mathbb{R}$ (for the flow) is a $C^1$-function satisfying the following conditions.[1]

1. $(dL/dt)(\varphi(t, \mathbf{x}))|_{t=0} \leq 0$ holds for each solution orbit $\{\varphi(t, \mathbf{x})\}$ through $\mathbf{x} \in U$.
2. $(dL/dt)(\varphi(t, \mathbf{x}))|_{t=0} = 0$ implies $\varphi(t, \mathbf{x}) \equiv \bar{\mathbf{x}} \in U$, where $\bar{\mathbf{x}}$ is an equilibrium of (1.1).

Lyapunov functions play central roles to gradient dynamical systems, which ensure that all trajectories behave so that Lyapunov functions decrease monotonously. Such behavior is expected locally around hyperbolic fixed points or general invariant sets. Once we construct Lyapunov functions locally around invariant sets, they provide information of local dynamics in terms of level sets of Lyapunov functions. In Definition 1.1, a Lyapunov function $L$ is, if exists, defined in an open set $U \subset \mathbb{R}^n$, but it does not tell us how large $U$ can be chosen. Furthermore, explicit constructions of Lyapunov functions themselves are not easy owing to nonlinearity of dynamical systems. For typical systems which possess Lyapunov functions with explicit forms, such systems themselves are fully determined by these functions, which are often called *gradient systems*; namely, $\frac{d}{dt}\mathbf{x} = \nabla L(\mathbf{x})$. Although there is an abstract result concerning with the existence of Lyapunov functions in more general sense, which is known as *Conley's Fundamental Theorem of Dynamical Systems*[2] (e.g. [1,2]), detections of the concrete form of $L$ and the concrete shape of $U$ near invariant sets remain open and depend on individual dynamical systems. In particular, we need to know information of *all* invariant sets in advance for constructing Lyapunov functions following Conley's theorem, which is not realistic for general systems and practical studies. Despite the great importance for dynamical systems, construction of Lyapunov functions in concrete systems remains open. As for preceding works about construction of Lyapunov functions with numerical computations, we leave several comments at the end of Introduction.

There are several preceding works for validating functionals like Lyapunov functions *rigorously* within explicit domains (e.g. [3–5]), many of which apply functionals called *cones* satisfying *cone conditions* (e.g. [6]) to understanding asymptotic behavior around invariant sets. Cones with cone conditions restrict the behavior of points in terms of differences between two points. In particular, these concepts describe stable and unstable manifolds of invariant sets as graphs of Lipschitzian (or smooth) functions. On the other hand, there is a preceding study for constructing Lyapunov functions in the sense of Definition 1.1 in explicitly given neighborhoods of hyperbolic equilibria [7]. There Lyapunov functions have very simple forms, and a sufficient condition for validating Lyapunov functions in given domains as well as hyperbolicity of equilibria for (semi)flows is proposed. In these two directions, there are several similarities. Firstly, functionals (cones, Lyapunov functions) describing asymptotic behavior have *quadratic forms* around equilibria or fixed points. Secondly, validations of functionals are done via negative definiteness of matrices associated with functionals. Thirdly, computer assisted analysis via *interval arithmetic* is applied to validating given quadratic forms being cones or Lyapunov functions in explicitly given domains.

This paper aims at a systematic procedure of Lyapunov functions around fixed points both for continuous and discrete dynamical systems by simple forms in given domains with computer assistance. This procedure gives us a general implementation of Lyapunov functions around fixed points, which can be applied to various dynamical systems including non-gradient-like systems, as indicated in preceding works (e.g. [3–5,8,9]).

Our central target function $L$ is a quadratic function as preceding works. The determination of domains $L$ being a Lyapunov function, called *Lyapunov domain*, consists of following two stages.

- **Stage 1: Negative definiteness of the matrix associated with** $dL/dt$.

For a domain containing an equilibrium, we verify the negative definiteness of specific matrix associated with $dL/dt(\varphi(t, \mathbf{x}))$ along solution orbits with computer assistance, which gives us a sufficient condition so that $L$ is a Lyapunov function in the domain. This stage is effective near equilibria.

- **Stage 2: Direct calculations of the negativity of** $dL/dt$ **along trajectories**.

For domains which do not contain equilibria, we calculate $dL/dt(\varphi(t, \mathbf{x}))$ directly with computer assistance and verify if it is negative. This stage is effective far from equilibria.

If one of such criteria passes for given domains, the quadratic function $L$ is validated as a Lyapunov function on the domain containing equilibria, even of saddle type. Combination of validations in these two stages with computer assistance gives us

[1] In several references, in addition to conditions in our definition, Lyapunov functions also require the condition "$L(x) \geq 0$ *for all* $x \in U$". This is often the case of functions around *asymptotically stable* equilibria. However, our focus includes *unstable* equilibria, and hence the situation $L(x) < 0$ typically happens around such equilibria. We do not thus assume $L(x) \geq 0$ in the definition of Lyapunov functions.

[2] The theorem is stated as follows: *dynamical systems on (whole) compact metric spaces whose chain recurrent sets are totally disconnected (i.e., strongly gradient-like dynamical systems) admit Lyapunov functions, defined in the whole space, in the sense that* $(dL/dt)(\mathbf{x}) = 0$ *implies that* $\mathbf{x}$ *is an equilibrium*. Fundamental properties of chain recurrent sets such as periodic orbits induce a generalized result: *dynamical systems on compact metric spaces admit Lyapunov functions, defined in the whole space, in the sense that* $(dL/dt)(\mathbf{x}) = 0$ *implies that* $\mathbf{x}$ *is on a chain recurrent component*, which gives one of generalized forms of Lyapunov functions. However, we only discuss the form of Lyapunov functions stated in Definition 1.1 because our focus is the local dynamics around equilibria or fixed points.