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Classical and potential symmetries for a generalized Fisher equation

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ABSTRACT

In this work, we consider a generalized Fisher equation and we have considered this equation from the point of view of the theory of symmetry reductions in partial differential equations. Generalizations of the Fisher equation are needed to more accurately model complex diffusion and reaction effects found in many biological systems. The reductions to ordinary differential equations are derived from the optimal system of subalgebras and new exact solutions are obtained. The potential system has been achieved from the complete list of the conservation laws. Potential symmetries, which are not local symmetries, are carried out for the generalized Fisher equation, these symmetries lead to the linearization of the equation by non-invertible mappings.

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1. Introduction

Fisher equations are commonly used in biology for population dynamics models and in bacterial growth problems as well as development and growth of solid tumors. The theory of reaction–diffusion waves begins in the 1930s with the works in population dynamics, combustion theory and chemical kinetics. At the present time, it is a well developed area of research which includes qualitative properties of traveling waves for the scalar reaction–diffusion equation and for system of equations, complex nonlinear dynamics, numerous applications in physics, chemistry, biology and medicine

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[1–3]. Reaction–diffusion equations are conventionally used in physical chemistry in order to describe concentration and temperature distributions. In this case, heat and mass transfer are described by the diffusion term while the reaction term describes the rate of heat and mass production.

The equation analyzed in this paper is a generalized Fisher equation

$$u_t = f(u) + \frac{1}{c(x)}\left(c(x)g(u)u_x\right)_x, \tag{1}$$

where $u(x, t)$ denotes the tumor cell density at location x and time t, being x and t the independent variables, $g(u)$ is the diffusion coefficient representing the active motility of cells depending on the variable u, $f(u)$ an arbitrary function and $c(x)$ an arbitrary function depending on the spatial variable x. In the particular case of $c(x) = 1$ and $g(u) = 1$, symmetry reductions and exact solutions were obtained using classical and nonclassical symmetries in [4]. Lie symmetry analysis of differential equations provides a powerful and fundamental framework to the exploitation of systematic procedures leading to the integration by quadrature of ordinary differential equations, to the determination of invariant solutions of initial and boundary value problems and to the derivation of conservation laws [5,6]. When $c(x) = 1$ but $g(u)$ and $f(u)$ are arbitrary functions, equation

$$u_t = f(u) + (g(u)u_x)_x \tag{2}$$

is known as the density dependent diffusion–reaction equation which is mentioned by J.D. Murray [2] to model the advance of an advantageous gene through a geographic region. In [7–9], we have derived conservation laws and exact solutions of several interesting particular cases of Eq. (1).

It is known that conservation laws play a significant role in the solution process of an equation or a system of differential equations and its applications to PDEs [10–18].

In [19], Anco and Bluman gave a general algorithmic method to find all conservations laws for evolution equations like Eq. (1). Many recent papers using this method have been published [20–22,7,23].

Local symmetries admitted by a nonlinear PDE are also useful to discover whether or not the equation can be linearized by an invertible mapping and construct an explicit linearization when one exists. A nonlinear scalar PDE is linearizable by an invertible contact (point) transformation if and only if it admits an infinite-parameter Lie group of contact transformations satisfying specific criteria [24–26]. An obvious limitation of group-theoretic methods based on local symmetries, in their utility for particular PDEs, is that many of these equations do not have local symmetries. It turns out that PDEs can admit nonlocal symmetries whose infinitesimal generators depend on integrals of the dependent variables in some specific manner. It also happens that if a nonlinear scalar PDE does not admit an infinite-parameter Lie group of contact transformations, it is not linearizable by an invertible contact transformation. However, most of the interesting linearizations involve non-invertible transformations; such linearizations can be found by embedding given nonlinear PDEs in auxiliary systems of PDEs [24]. Krasil'shchik and Vinogradov [27] gave criteria which must be satisfied by nonlocal symmetries of a PDE when realized as local symmetries of a system of PDEs which covers the given PDE.

In [24,25], Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by $R\{x, t, u\}$ in a conserved form, a related system denoted by $S\{x, t, u, v\}$ with potentials as additional dependent variables is obtained. Any Lie group of point transformations admitted by $S\{x, t, u, v\}$ induces a symmetry for $R\{x, t, u\}$; when at least one of the generators of the group depends explicitly on the potential, then the corresponding symmetry is neither a point nor a Lie–Bäcklund symmetry. These symmetries of $R\{x, t, u\}$ are called potential symmetries. The nature of potential symmetries allows one to extend the uses of point symmetries to such nonlocal symmetries. In particular:

(i) Invariant solutions of $S\{x, t, u, v\}$ yield solutions of $R\{x, t, u\}$ which are not invariant solutions for any local symmetry admitted by $R\{x, t, u\}$.
(ii) If $R\{x, t, u\}$ admits a potential symmetry leading to the linearization of $S\{x, t, u, v\}$ then $R\{x, t, u\}$ is linearized by a non-invertible mapping.

In [9], we have constructed conservation laws for Eq. (1). These conservation laws were derived by using a conservation theorem due to Ibragimov, as well as the multiplier method of Anco and Bluman [19].

In this work, we consider a particular case of Eq. (1) when $c(x) = k_1e^{rx}$, $g(u) = k_2u^{-2}$ and $f(u) = k_3u$ with k_1, k_2, k_3 and r arbitrary constants

$$u_t = k_3u + \frac{1}{k_1e^{rx}}\left(k_1e^{rx}k_2u^{-2}u_x\right)_x. \tag{3}$$

By using the symmetry generators previously derived in [9], we perform the optimal system of subalgebras, the similarity reductions, and we find for exact solutions of physical and chemical interest.

For Eq. (3) it happens that some of the associated conserved systems admit symmetries that yield to potential symmetries. These symmetries are realized as local symmetries of a related auxiliary system, and lead to the construction of corresponding invariant solutions, as well as to the linearization of the equation by non-invertible mappings.

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