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The implicit midpoint method for the modified anomalous sub-diffusion equation with a nonlinear source term

Xuenian Cao *, Xianxian Cao, Liping Wen

School of Mathematics and Computational Science, Xiangtan University, Hunan 411105, China

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A B S T R A C T

In this paper, the implicit midpoint method is used to solve the semi-discrete modified anomalous sub-diffusion equation with a nonlinear source term, and the weighted and shifted Grünwald–Letnikov difference operator and the compact difference operator are applied to approximate the Riemann–Liouville fractional derivative and space partial derivative respectively, then the new numerical scheme is constructed. The stability and the convergence of this method are analyzed. Numerical experiment demonstrates the high accuracy of this method and confirm our theoretical results.

1. Introduction

There has been increasing interest in the description of many phenomenon and processes by means of equations involving fractional derivatives over the last decades [1–8]. Among these applications, the anomalous sub-diffusion equation has attracted considerable attention. Recently there are models that have been proposed to describe process that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a diffusion operator with a nonlinear source term [9,10]

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} = (A_0 D_t^{1-\alpha} + B_0 D_t^{1-\beta})\frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t),x,t), \quad 0 \le x \le S, \qquad 0 \le t \le T \\
&u(0,t) = \varphi_1(t), \qquad u(S,t) = \varphi_2(t), \quad 0 \le t \le T \\
&u(x,0) = \psi(x), \quad 0 \le x \le S
\end{aligned} \tag{1}$$

where $0 < \alpha, \beta < 1$, A, B are positive constants, the symbol ${}_0D_t^{1-\gamma} u(x,t)$ denotes the Riemann–Liouville fractional derivative operator, which is defined by

$${}_0D_t^{1-\gamma} u(x,t) = \frac{1}{\Gamma(\gamma)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\eta)}{(t-\eta)^{1-\gamma}} d\eta, \quad \gamma = \alpha, \beta,$$

* Corresponding author.
E-mail address: cxn@xtu.edu.cn (X. Cao).

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where $\Gamma(\cdot)$ is the gamma function, $f(u,x,t)$ satisfies the Lipschitz condition with respect to u:

$$|f(u,x,t)-f(\upsilon,x,t)|\le L|u-\upsilon|,\quad \forall u,\upsilon$$

here L is Lipschitz constant.

Furthermore, we assume that the problem (1) has a unique sufficiently smooth exact solution $u(x,t)$.

Much work have been done on developing numerical methods for solving the modified anomalous sub-diffusion equation, see e.g. [11–16,9,17,10,18–20]. However the numerical methods and their numerical analysis are still too little for solving problem with a nonlinear source term (1). Liu et al. [10] proposed an implicit difference method, and proved the stability and convergence using the energy method, and the convergence order of the method is $O(\tau+h^2)$. Liu et al. [9] proposed a semi-discrete approximation and a full discrete finite element approximation, and proved the stability and convergence of the proposed methods. Mohebbi et al. [18] obtain fully discrete implicit scheme by applying the Grünwald–Letnikov and the compact difference operator to discrete Riemann–Liouville fractional derivative and space partial derivative respectively, and proved that the compact difference scheme is unconditionally stable and convergent by Fourier analysis. The convergence order of the method is $O(\tau+h^4)$. Li et al. [17] proposed a numerical method which used the weighted and shifted Grünwald–Letnikov difference operator and the compact difference operator to approximate the Riemann–Liouville fractional derivative and the space partial derivative respectively, and used the second order backward difference formula to solve the semi-discrete system obtained by discretizing space variable. The convergence order of the method is shown to be $O(\tau^2+h^4)$. This method have higher accuracy on time variable. But the implementation of the algorithms needs two initial starting information. Beside a given initial value, another starting information needs to compute by other numerical method. So it may affect the accuracy of the algorithm. The objective of this paper is to try to use the implicit midpoint formula to achieve a new numerical method with high accuracy.

The outline of this paper is as follows. In Section 2, the numerical method for the modified anomalous sub-diffusion equation is given. Then, in Section 3, stability and convergence analysis are discussed, respectively. Section 4 is used to present numerical results, comparing the fixed stepsize implementation on a test problem. Numerical experiment shows that the proposed method has high accuracy and efficiency for solving the modified anomalous sub-diffusion equation.

2. Numerical method

In this paper, we assume that $u(x,t)\in U(\Omega)$, where

$$U(\Omega)=\left\{u(x,t)\mid \frac{\partial^6 u(x,t)}{\partial x^6},\frac{\partial^3 u(x,t)}{\partial x^2\partial t},\frac{\partial^3 u(x,t)}{\partial t^3}\in C(\Omega)\right\},$$

whereas $\Omega=\{(x,t)|0\le x\le S,\ 0\le t\le T\}$.

For the space interval $[0,S]$ and time interval $[0,T]$, we choose the grid points as follows $x_j=jh,\ j=0,1,\dots,M,\ t_n=n\tau,\ n=0,1,\dots,N$, where $h=\frac{S}{M}$ denotes spatial step size, $\tau=\frac{T}{N}$ denotes time stepsize. The exact solution and numerical solution at the point (x_j,t_n) are denoted by $u(x_j,t_n)$ and u_j^n respectively.

Lemma 2.1 *([21]). If $u(x,t)\in U(\Omega)$, then*

$$\left(1+\frac{1}{12}\delta_x^2\right)\frac{\partial^2 u(x_j,t_n)}{\partial x^2}=\frac{\delta_x^2 u(x_j,t_n)}{h^2}+O(h^4) \tag{2}$$

where $\delta_x^2 u(x_j,t_n)=u(x_{j-1},t_n)-2u(x_j,t_n)+u(x_{j+1},t_n)$.

Lemma 2.2 *([22,20]). Let x be a grid point, $u(x,t)\in L^1(\mathbb{R})$, ${}_{-\infty}D_t^{\alpha+2}u(x,t)$ and its Fourier transform belong to $L^1(\mathbb{R})$, and define the weighted and shifted Grünwald–Letnikov difference operator by*

$$D_{\tau,p,q}^{\alpha}u(x,t)=\frac{\alpha-2q}{2(p-q)}A_{\tau,p}^{\alpha}u(x,t)+\frac{2p-\alpha}{2(p-q)}A_{\tau,q}^{\alpha}u(x,t),$$

then we have

$$D_{\tau,p,q}^{\alpha}u(x,t)={}_{-\infty}D_t^{\alpha}u(x,t)+O(\tau^2),\quad t\in\mathbb{R},$$

where p,q are integers and $p\ne q$, $A_{\tau,r}^{\alpha}u(x,t)$ is the Grünwald–Letnikov approximation to the Riemann–Liouville fractional derivative on variable t by

$$A_{\tau,r}^{\alpha}u(x,t)=\tau^{-\alpha}\sum_{k=0}^{\infty}g_k^{(\alpha)}u(x,t-(k-r)\tau),\quad r=p,q,$$