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Minimal faithful upper-triangular matrix representations for solvable Lie algebras

M. Ceballos [a], J. Núñez [b,*], Á.F. Tenorio [c]

[a] Dpto. de Matemáticas e Ingeniería, Universidad Loyola Andalucía, Campus Palmas Altas C/ Energía Solar 1, Ed. E, 41014-Seville, Spain

[b] Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad de Sevilla, Calle Tarfia, s/n, 41012-Seville, Spain

[c] Dpto. de Economía, Métodos Cuantitativos e Historia Económica, Escuela Politécnica Superior, Universidad Pablo de Olavide, Ctra. Utrera km. 1, 41013-Seville, Spain

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ABSTRACT

The existence of matrix representations for any given finite-dimensional complex Lie algebra is a classic result on Lie Theory. In particular, such representations can be obtained by means of an isomorphic matrix Lie algebra consisting of upper-triangular square matrices. Unfortunately, there is no general information about the minimal order for the matrices involved in such representations. In this way, our main goal is to revisit, debug and implement an algorithm which provides the minimal order for matrix representations of any finite-dimensional solvable Lie algebra when inserting its law, as well as returning a matrix representative of such an algebra by using the minimal order previously computed. In order to show the applicability of this procedure, we have computed minimal representatives not only for each solvable Lie algebra with dimension less than 6, but also for some solvable Lie algebras of arbitrary dimension.

1. Introduction

Representation Theory of Lie algebras has broad applications to the analysis of continuous symmetries in Mathematics and Physics. More concretely, in Mathematics, the classification of Lie groups reveals symmetries in differential equations. With respect to Physics, representation theory yields natural connections between representation of Lie algebras and the properties of elementary particles.

Ado's Theorem states that given a finite-dimensional complex Lie algebra $\mathfrak{g}$, there exists a matrix algebra isomorphic to $\mathfrak{g}$ (see [1] for the classical proof and [2] for a very short alternative). In this way, every finite-dimensional complex Lie algebra can be represented as a Lie subalgebra of the complex general linear algebra $\mathfrak{gl}(n; \mathbb{C})$, of complex $n \times n$ matrices, for some $n \in \mathbb{N}$.

This paper focuses on the Lie algebra $\mathfrak{h}_n$, of $n \times n$ upper-triangular matrices, and the existence of an isomorphism between an arbitrary given finite-dimensional solvable Lie algebra $\mathfrak{g}$ and a subalgebra of $\mathfrak{h}_n$ for some $n \in \mathbb{N}$ (proved, for example, in [3, Proposition 3.7.3]). Taking into consideration this result, one can be interested in knowing which $n \in \mathbb{N}$ is the minimum such that $\mathfrak{h}_n$ admits a subalgebra being isomorphic to $\mathfrak{g}$; in other words, finding the minimal faithful representation of $\mathfrak{g}$ by means of upper-triangular matrices.

* Corresponding author.
E-mail addresses: mceballos@uloyola.es (M. Ceballos), jnvaldes@us.es (J. Núñez), aftenorio@upo.es (Á.F. Tenorio).

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Although many authors have studied the notion of minimal dimension $\mu(\mathfrak{g})$ for representing a Lie algebra $\mathfrak{g}$ (see Burde [4], for instance), most of literature has used $\mathfrak{g}$-modules for faithful representations instead of Lie algebras $\mathfrak{h}_n$ (as subclass of $\mathfrak{g}$-modules). Therefore, the value of $\mu(\mathfrak{g})$ is less than or equal to the dimension to be computed in this paper. Moreover, even when matrix representations were computed (e.g. Ghanam et al. [5] for low-dimensional nilpotent Lie algebras), the minimality of such representations was not considered and non-minimal representations were obtained.

The interest in these faithful representations is motivated, among other issues, by problems from Geometry and Topology. For example, Milnor [6] and Auslander [7,8] studied generalizations of crystallographic groups in relation to this minimal value for matrix representations. Another motivation is based on the following result: if $\mathfrak{g}$ is the Lie algebra of a given Lie group G admitting a left-invariant affine structure, then the minimal dimension of its faithful representations is $\mu(\mathfrak{g}) \leq n+1$, where n is the dimension of $\mathfrak{g}$.

Several papers throughout the literature deal with matrix representations of solvable Lie algebras, focusing on the subclass of nilpotent Lie algebras. In this sense, representatives of minimal faithful unitriangular matrix representations were explicitly computed in Benjumea et al. [9] for given nilpotent Lie algebras by application of an algorithmic procedure. However, complete lists of such representations for nilpotent Lie algebras of dimension less than 6 and filiform Lie algebras of dimension less than 9 were given later in [10,11], respectively. Additionally, Núñez and Tenorio [12] continued with this research and gave the outlines of an algorithmic procedure to compute explicitly representatives of the minimal faithful matrix representation for solvable Lie algebras by using Lie algebras $\mathfrak{h}_n$, giving some examples of application by hand. This procedure adapted that given in Benjumea et al. [9], but neither the algorithm was completely debugged nor implementations were carried out and run.

The main goal of the current paper is to advance in the above-mentioned research by debugging and implementing the algorithm sketched in [12] in order to automate the computation of minimal faithful matrix representations for a given solvable Lie algebra starting from its law. As an application, we have also computed representations for each solvable Lie algebra of dimension less than 6 as well as for others of higher dimension. To do so, we have used the classifications given by Mubarakzyanov and Turkowski (see [13–16]).

The structure of this paper is the following: In Section 2, some theoretical background on Lie Theory is recalled in order to be applied in later sections. Thereupon, Section 3 revisits the algorithmic procedure sketched in [12] to obtain a minimal faithful representation of a given solvable Lie algebra by means of upper-triangular matrices, incorporating a formulation of the algorithm which can be dealt computationally with its implementation in MAPLE 12. To shorten the paper length, we only show the application of the above-mentioned computational method to two algebras in Section 4. Just afterwards, Section 5 shows a list with explicit representatives of minimal faithful matrix representations for solvable Lie algebras of dimension less than 6. Some papers related with this topic can be consulted in [17–21] and the references therein.

2. Preliminaries

An overall review on Lie algebras can be consulted in [3]. The present section only recalls some definitions and results about Lie algebras which we are applying later. From here on, we only consider finite-dimensional Lie algebras over the complex number field $\mathbb{C}$.

Given a Lie algebra $\mathfrak{g}$, its *derived series* is defined as follows

$$\mathcal{C}_1(\mathfrak{g}) = \mathfrak{g},\ \mathcal{C}_2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}],\ \mathcal{C}_3(\mathfrak{g}) = [\mathcal{C}_2(\mathfrak{g}), \mathcal{C}_2(\mathfrak{g})], \dots, \mathcal{C}_k(\mathfrak{g}) = [\mathcal{C}_{k-1}(\mathfrak{g}), \mathcal{C}_{k-1}(\mathfrak{g})], \dots. \tag{1}$$

Additionally, the Lie algebra $\mathfrak{g}$ is said to be *solvable* if there exists a natural integer m such that $\mathcal{C}_m(\mathfrak{g}) \equiv 0$. The solvability index of $\mathfrak{g}$ is precisely the value of $m \in \mathbb{N}$ such that $\mathcal{C}_m(\mathfrak{g}) = 0$ and $\mathcal{C}_{m-1}(\mathfrak{g}) \neq 0$.

The relation between the derived series of a given Lie algebra $\mathfrak{g}$ and that of a Lie subalgebra is given as follows.

Proposition 1. *If $\mathfrak{h}$ is a Lie subalgebra of a given Lie algebra $\mathfrak{g}$, then $\mathcal{C}_k(\mathfrak{h}) \subseteq \mathcal{C}_k(\mathfrak{g})$, for all $k \in \mathbb{N}$.*

Given $n \in \mathbb{N}$, the complex solvable Lie algebra $\mathfrak{h}_n$ consists of $n \times n$ upper-triangular matrices; i.e. its vectors are expressed as

$$h_n(x_{r,s}) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{nn} \end{pmatrix}, \quad \text{with } x_{r,s} \in \mathbb{C},\ \text{for } 1 \leq r \leq s \leq n. \tag{2}$$

The Lie algebra $\mathfrak{h}_n$ has a basis $\mathcal{B}_n$ consisting of vectors $X_{i,j} = h_n(x_{r,s})$ with $1 \leq i \leq j \leq n$ and such that

$$x_{r,s} = \begin{cases} 1, & \text{if } (r,s) = (i,j), \\ 0, & \text{if } (r,s) \neq (i,j). \end{cases} \tag{3}$$

The dimension of $\mathfrak{h}_n$ is $\frac{n(n+1)}{2}$ and the nonzero brackets with respect to basis $\mathcal{B}_n$ are

$$[X_{i,j}, X_{j,k}] = X_{i,k}, \quad \forall\, 1 \leq i < j < k \leq n; \tag{4}$$