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An algorithm to compute invariant sets for third-order switched systems

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ABSTRACT

We study invariant sets for switched systems. We focus on a class of third-order switched systems. In this class of switched systems we solve the problem of finding an invariant set for a switching law. For each switched system we provide the invariant set and the switching law, in addition, the invariant set is a polyhedral cone and the switching law is switching on the boundary. To accomplish this we reduce the problem to a simplified system. We provide a procedure calculating the invariant set. The method is based on calculating an invariant set for a simplified system and then transformating the invariant set to the original system. We illustrate this method with an example.

1. Introduction

A switched linear system is a dynamical system which consists of several subsystems, and one of them is active each time. There are two possibilities of studying properties of a switched system, one way is when a property holds for every switching law and other way is when a property holds for at least one switching law and in this last case the problem is to design a switching law. For instance, stability is studied in these two ways, see the survey paper [1].

In this paper we only consider continuous switched system, although there are examples and papers on discrete switched systems [2]. A wide number of properties for dynamical systems have been studied for switched system as well. For example controllability and reachability [3], stability and stabilization [1,4], optimization [5]. See the book [6] to find others problems for switched systems.

We study the problem of invariant set for switched systems. Invariant sets for dynamical system have been studied for example in [7]. Invariant set in the scope of switched system has been given in the paper [8] but in the context of arbitrary switching law. However we study when a switched system has an invariant set for some switching law, so we treat with the problem of design a switching law.

In the next section we describe the problem of invariant set for switched systems. We focus on a class of third-order systems where we are able to solve the problem. The problem is solved in Section 3, to accomplish this we reduce the problem to a simplified case. In this simplified case we give an invariant set. Finally we transform the invariant set to the original system. In Section 4 we give an example and illustrate how to calculate the invariant set. In Section 5 we provide the conclusions.

2. Problem description

A switched linear system is a system with several subsystems

$$\dot{x} = A_k x, \quad k = 1, \ldots, M, \tag{1}$$

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where M is the number of subsystems and $A_1, \dots, A_M$ are $n \times n$ matrices. Only one subsystem is active each time, the decision of which subsystem is active is given by a switching law. A switching law says when the active subsystem changes, this decision of switching to other subsystem can be given in terms of time or state. For our purposes, the switching law depends on the state. Then, a switching law $\sigma : \mathbb{R}^n \to \{1, \dots, M\}$. Given a switching law σ the system is

$$\dot{x} = A_{\sigma(x)}x, \tag{2}$$

and we denote $\phi(\cdot; x_0, \sigma)$ the solution of (2) with initial condition $x(0) = x_0 \in \mathbb{R}^n$.

The following definition is the fundamental property of this paper.

Definition 2.1. A set $S \subset \mathbb{R}^n$ is an invariant set for the switched system (1) if there exists a switching law σ such that if $x_0 \in S$ then $\phi(t; x_0, \sigma) \in S$ for each $t \geq 0$.

For second-order switched system with two subsystems, if A_1, A_2 are 2×2 matrices corresponding to each subsystem. In case that A_1 and A_2 have complex eigenvalues (two conjugate complex and non-real eigenvalues) and turn clockwise and counter-clockwise, respectively, then it is easy to check that every cone in $\mathbb{R}^2$ is an invariant set, by switching on the boundary of the cone. For more details see [9].

We desire a similar behavior in third-order, but the same argumentation it does not work for third-order. Although an invariant set can be given for a certain class in third-order.

We consider a third-order switched systems with three subsystems

$$\dot{x} = A_k x, \quad k = 1, 2, 3, \tag{3}$$

where A_1, A_2, A_3 are 3×3 matrices. We are interested in switched systems with the following assumptions.

Assumption 2.2. Each A_k, $k = 1, 2, 3$, has complex eigenvalues,i.e. the numbers λ_k, $a_k + b_k i$, $a_k - b_k i$ are the eigenvalues of A_k with $b_k \neq 0$.

Assumption 2.3. Let $v_k \in \mathbb{R}^3$ be an eigenvector of A_k associated to the real eigenvalue λ_k, $k = 1, 2, 3$, i.e. v_k is a non-zero vector such that $A_k v_k = \lambda_k v_k$, then v_1, v_2, v_3 are linear independent vectors.

Our goal is to show that there is an invariant set for every switched system with Assumptions 2.2 and 2.3.

3. Invariant set for third-order switched systems

We deal with our problem in several steps. First we show the relation between the class of third-order switched system above-named and other simplified class of switched system. Then we give an invariant set for this new simplified switched system. Finally we transform the invariant set for simplified switched system to the invariant set for a general class of switched systems.

3.1. Simplification of the class of third-order switched systems

The following proposition is useful for studying an invariant set for other switched systems, we will use it in order to get a simplified switch system where we are able to provide an invariant set.

Proposition 3.1. *Let P be a $n \times n$ non-singular matrix and $S \subset \mathbb{R}^n$ be a closed set. The following statements are equivalent*

1. *$P(S) \subset \mathbb{R}^n$ is an invariant set for the switched system*
$$\dot{x} = A_{\sigma \circ P^{-1}} x,$$
2. *$S \subset \mathbb{R}^n$ is an invariant set for the switched system*
$$\dot{y} = P^{-1} A_\sigma P y$$

where we denote $P(S) = \{Py : y \in S\}$ and $\sigma \circ P^{-1}(x) = \sigma(P^{-1}x)$ for $x \in \mathbb{R}^n$.

Proof. We only prove one implication because the other one can be proven in a similar way. We will prove that the second statement implies the first one by contraposition. Suppose that there is an initial condition $x_0 \in P(S)$ and a time $t' > 0$ such that $x(t') \notin P(S)$, then we define

$$T = \sup\{t \geq 0 : x(\tau) \in P(S), 0 \leq \tau \leq t\}.$$

We consider $x_1 = x(T)$, and since the solution is continuous and S is a closed set, then $x_1 \in P(S)$. Also, for every $\delta > 0$ there is a time $0 < t < \delta$ such that $x(T + t) \notin P(S)$. For the initial condition $y_0 = P^{-1}x_1$, it turns out that $y_0 \in P$ and