ARTICLE IN PRESS

Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On compact representations for the solutions of linear difference equations with variable coefficients

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ARTICLE INFO

Article history:
Received 30 September 2015
Received in revised form 12 January 2016

MSC:
15A99
39A10

Keywords:
Enumerative combinatorics
Hessenbergian
Linear difference equation
Nested sum

ABSTRACT

A comprehensive treatment on compact representations for the solutions of linear difference equations with variable coefficients, of both nth and unbounded order, is presented. The equivalence between their celebrated combinatorial and determinantal representations is considered. A corresponding representation is proposed using determined nested sums of their variable coefficients. It makes explicit all the sum of products involved in the previous representations of such solutions. Some basic applications are also illustrated.

1. Introduction

Compact representations for the solutions of linear difference equations with variable coefficients, LDE for short, of both finite and unbounded order are of interest in many branches of science and engineering; see e.g. [1]. Some approaches for representing the solutions of LDE have been introduced in the literature. Among these, most noteworthy have been the determinantal representations [2,3], and the combinatorial one [4]. The determinantal representation uses hessenbergians, determinants of Hessenberg submatrices (see e.g. [5,6]) of a single solution matrix. The combinatorial representation is based on determined combinations of sums of products of their variable coefficients.

The nested sums have resulted to be useful for obtaining explicit representations of complex combinatorial formulas, e.g. with binomial, Gaussian binomial, or Stirling-like coefficients [7]. These nested structures have been applied on the expansion of transcendental functions and multiscale multiloop integrals [8], on orthogonal polynomials, linear non-autonomous area-preserving maps, representations for the inverses of tridiagonal matrices, and also on continued fractions; see e.g. [9,10] and the references therein. Relative to LDE, the nested sums are suitable for representations of the solutions of parameterized LDE [11], and of the second-order LDE [9].

Our purpose is twofold. First, it is natural to consider the equivalence of the hessenbergian representation for the solutions of LDE [2,3], with respect to the combinatorial one [4]. Furthermore, it is also of use to establish simpler representations of such solutions. The suitability of the nested sums regarding more explicit representation for the solutions of LDE will be pointed up.

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http://dx.doi.org/10.1016/j.cam.2016.02.049

Please cite this article in press as: J. Abderramán Marrero, V. Tomeo, On compact representations for the solutions of linear difference equations with variable coefficients, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.02.049

The outline is as follows. A comprehensive treatment on compact representations for the solutions of LDE of both nth and unbounded order is discussed in Section 2. The equivalence of the hessenbergian representation respecting the combinatorial one is checked. A more explicit representation for the solutions of LDE of unbounded order based on nested sums is proposed in Section 3. Although it presents a more involved problem, a representation for the nth order LDE with nested sums is also introduced. We illustrate in Section 4 with basic examples of their potential applications. Thus compact representations based on nested sums for hessenbergians, inverses of triangular matrices, multinomial distribution, and the Roger–Szegö polynomials, are managed.

2. Representations for the solutions of LDE and their equivalence

The equivalence between the hessenbergian [2,3] and the combinatorial representation [4] is checked. With this aim, we begin focusing on the notation and results about the hessenbergian representation given in [2].

2.1. LDE of unbounded order

Following [2], a LDE of unbounded order can be formulated as

$$\sum_{i=1}^{k} p(k,i)y_i = f(k), \tag{1}$$

the coefficients $p(k,i)$, and the nonhomogeneous terms $f(k)$, are known functions. Here the coefficients $p(k,k)$ satisfying $p(k,k) \neq 0$, for every $k \in \mathbb{Z}^+$. Since y_n depends only on $y_1, y_2, \ldots, y_{n-1}$, the first n equations allow us to attain y_n. Indeed, given the (infinite) lower unreduced Hessenberg matrix

$$\mathbf{R} = \begin{bmatrix} f(1) & p(1,1) & 0 & 0 & \cdots \\ f(2) & p(2,1) & p(2,2) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{2}$$

a representation of the solutions of (1) using hessenbergians [5,6] is

$$y_n = \frac{(-1)^{n-1}}{\prod_{i=1}^{n} p(i,i)} \det \mathbf{R}_n. \tag{3}$$

The finite matrix $\mathbf{R}_n$,

$$\mathbf{R}_n = \begin{bmatrix} f(1) & p(1,1) & 0 & \cdots & 0 \\ f(2) & p(2,1) & p(2,2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ f(n-1) & p(n-1,1) & p(n-1,2) & \cdots & p(n-1,n-1) \\ f(n) & p(n,1) & p(n,2) & \cdots & p(n,n-1) \end{bmatrix}, \tag{4}$$

is the nth section of the matrix $\mathbf{R}$. Using elementary properties of the determinants, and defining the ratios $x_i = \frac{f(i)}{p(i,i)}$, $b_{ki} = -\frac{p(k,i)}{p(k,k)}$, formula (3) yields

$$y_n = \det \mathbf{R}_n^* = |\mathbf{R}_n^*|, \tag{5}$$

with the matrix

$$\mathbf{R}_n^* = \begin{bmatrix} x_1 & -1 & 0 & \cdots & 0 \\ x_2 & b_{21} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ x_{n-1} & b_{n-1,1} & b_{n-1,2} & \cdots & -1 \\ x_n & b_{n1} & b_{n2} & \cdots & b_{n,n-1} \end{bmatrix}. \tag{6}$$

Furthermore, expanding the hessenbergian (5) by the first column of the matrix $\mathbf{R}_n^*$,

$$y_n = \sum_{i=1}^{n} C_{i,1}^{(n)} x_i, \tag{7}$$

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