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On the exponential decay of waves with memory

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ABSTRACT

In this paper we consider general linear damped wave equations with memory. We establish energy estimates that under the assumption of exponentially bounded kernels, induce exponential decaying solutions. Numerical waves that mimic their continuous counterpart are also introduced using a finite element approach.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. This paper is concerned with the study of the decay of the solutions of the following damped wave equation with memory

$$\rho u''(t) + cu'(t) + \mathcal{A}u(t) = \int_0^t Ker\,(t-s;\tau)\,\mathcal{B}u(s)ds + f(t), \quad t \in \mathbb{R}^+. \tag{1}$$

In (1) $u : \overline{\Omega} \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ and, for $t \in \mathbb{R}_0^+$, $u(\cdot, t)$ can be seen as a function defined from $\overline{\Omega}$ into $\mathbb{R}$ that is denoted by $u(t)$, c is a function depending only on spatial variables and accounts for the damping of the wave, Ker denotes a function, called the *memory kernel*, that depends on a parameter $\tau > 0$, f denotes a source term and $\mathcal{A}$ and $\mathcal{B}$ are second order differential operators. Eq. (1) is completed with homogeneous Dirichlet boundary conditions and the following initial conditions

$$\begin{cases} u(0) = u_0, \\ u'(0) = u_1. \end{cases} \tag{2}$$

This type of differential problem arises in many contexts, such as modelling the displacement of materials with viscoelastic properties. Indeed, let u denote the displacement of the material, f an external force being applied to the material and $\boldsymbol{\sigma}$ the stress tensor associated. Newton's second law states that

$$\rho u''(t) = \nabla \cdot \boldsymbol{\sigma}(t) + f(t), \tag{3}$$

where ρ is the density of the material. Usually, the relation considered between the stress tensor $\boldsymbol{\tau}$ and the strain tensor $\boldsymbol{\epsilon}$ is

$$\boldsymbol{\sigma}(t) = \mathbf{D}\boldsymbol{\epsilon}(t) \tag{4}$$

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where $\mathbf{D}$ is an elastic tensor. Assuming that the components of the strain and the displacement satisfy

$$\boldsymbol{\epsilon}(t) = \frac{1}{2}\left(\nabla u(t) + \nabla u(t)^t\right),$$

relation (4) accounts for a fickian type effect. However, if we assume that the material has viscoelastic properties modelled by a Maxwell–Wiechert model and assume the following constitutive equation

$$\boldsymbol{\sigma}(t) = E(0)\mathbf{D}\epsilon(t) - \int_0^t \frac{\partial}{\partial s}E(t-s)\mathbf{D}\boldsymbol{\epsilon}(s)\,ds,$$

where

$$E(t) = E_0 + \sum_{i=1}^{N} E_i e^{-\alpha_i t}$$

and E_0 is the Young modulus of the spring arm, E_i, $i = 1, \ldots, N$, are the Young modulus of the Maxwell arms and $\alpha_i = \frac{E_i}{\mu_i}$, $i = 1, \ldots, N$, being μ_i, $i = 1, \ldots, N$, their associated viscosities, then from (3) we obtain for the displacement the following second order integro-differential equation

$$\rho u''(t) - \nabla \cdot \left(\frac{1}{2}E(0)\mathbf{D}(\nabla u(t) + \nabla u(t)^t)\right) = -\int_0^t Ker\ (t-s)\,\nabla \cdot \left(\mathbf{D}(\nabla u(s) + \nabla u(s)^t)\right)\,ds + f(t),$$

with $Ker\ (t) = \frac{1}{2}\sum_{i=1}^{N} E_i \alpha_i e^{-\alpha_i t},\ t \geqslant 0$.

Equations of type (1) have already been introduced in the literature, see [1–3], to model viscoelastic physical phenomena.

Let us consider the classical wave equation with homogeneous Dirichlet boundary conditions. It is well known that the energy of its solution (which is the sum of kinetics and potential energies) is conserved in time. If a damping effect is added then it can be shown that such energy decreases exponentially in time (see Section 3). In certain scenarios, the wave equation with a memory term can be seen as a singular perturbation of the diffusion equation with memory. The solution of this last equation has, in several cases, an energy that goes to zero exponentially (see Section 4).

A question that naturally arises is which conditions on the memory kernels lead to the same energy behaviour for the solution of the wave equation or its generalization in presence of a memory effect. This problem has been object of research in recent years and will be addressed in the present paper.

The study of qualitative properties of partial differential problems defined by equations of type (1) was presented for instance in [4–11]. However, these works deal essentially with energy estimates for the case when $\mathcal{A}$ and $\mathcal{B}$ represent the Laplace operator, combined with exponential or polynomial decaying kernels. For example, in [6], the authors studied the energy decay for a wave equation with nonlinear boundary damping. Also, in [9], acoustic boundary conditions were considered and the authors established energy decrease results when the kernel function does not necessarily decay exponentially. Similar results were obtained in [11] considering homogeneous Dirichlet boundary conditions but imposing weak assumptions on the memory kernel. The study of the decay of the solution of systems of wave equations has also been addressed in [4,10]. In [4] the authors established energy decreasing results for systems of two linear wave equations with memory with homogeneous Dirichlet boundary conditions with kernels exponentially dominated.

Energy decreasing results for quasilinear wave equations with memory were considered in [5,8]. In the first paper the authors consider a nonlinear reaction term and a wave equation where the coefficient of the second derivative depends on the solution was introduced in the second paper. Wave equations with memory as singular perturbations of nonfickian diffusion equations with memory have also been studied. Without being exhaustive we mention [12–15].

This work aims at establishing energy estimates (and show their exponential decay) for several variants of Eq. (1). This shall be accomplished in the case $\mathcal{A}$ and $\mathcal{B}$ represent the Laplace operator, but also in the more general setting as presented by (1), always under the assumption that the memory kernel decays exponentially.

The paper is organized as follows: we start in Section 2 by introducing the functional context necessary for the development of the energy estimates, as well as some properties of the kernels. In Section 3 we start by considering the wave equation with no memory ($\mathcal{A} = \mathcal{B} = -\Delta$) and review classical estimates for this case. In Section 4, we explore the case where the coefficient of the second time derivative vanishes, that is, the wave equation with memory is replaced by the diffusion equation with memory that is usually used to model diffusion phenomena (characterized by a nonfickian behaviour). We show that under suitable assumptions on the parameters of the equation, exponential decay of the waves is obtained. Damped wave equations with memory is the object of study of Section 5. In this section we introduce a new energy functional that is obtained from the classical one adding a new term induced by its memory character. We establish conditions that lead to the exponential decreasing of such energy functional. To measure the deviation of the gradient of the solution and its evolution in time, a new term is added to the energy functional under analysis. For this new energy functional we prove also its exponential decreasing. The techniques used to obtain these estimates (as well as the estimates themselves) are the motivation of a new energy functional definition for the first equation to be explored in the coming section. Indeed, similar results

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