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A new bi-parametric family of temporal transformations to improve the integration algorithms in the study of the orbital motion

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ABSTRACT

One of the fundamental problems in celestial mechanics is the study of the orbital motion of the bodies in the solar system. This study can be performed through analytical and numerical methods. Analytical methods are based on the well-known two-body problem; it is an integrable problem and its solution can be related to six constants called orbital elements. To obtain the solution of the perturbed problem, we can replace the constants of the two-body problem with the osculating elements given by the Lagrange planetary equations. Numerical methods are based on the direct integration of the motion equations. To test these methods we use the model of the two-body problem with high eccentricity.

In this paper we define a new family of anomalies depending on two parameters that includes the most common anomalies. This family allows one to obtain more compact developments to be used in analytical series and also to improve the efficiency of numerical methods because it defines a more suitable point distribution with the dynamics of the two-body problem.

1. Introduction

One of the most important problems in celestial mechanics is the study of the orbital motion around a central body. This problem includes the planetary theories and the study of the motion of artificial satellites around the Earth. The problem of the motion of a planet around the Sun or an artificial satellite around the Earth can be modelized by means of the second order differential equations

$$\frac{d^2\vec{r}}{dt^2} = -\mu\frac{\vec{r}}{r^3} - \vec{\nabla}U + \vec{F} \tag{1}$$

where $\vec{r}$ is the radius vector referred to the primary (the Sun in the case of a planet or the Earth in the case of an artificial satellite), G the gravitational constant, m the mass of the primary, m' the mass of the secondary (planet or artificial satellite), $\mu = G(m + m')$ the spaceflight constant, U the potential that induces the conservative perturbation forces such as the gravitational forces; and $\vec{F}$ represents non-conservative disturbing forces such as atmospheric friction, radiation pressure, the solar wind, and others. The most important motion in the solar system can be described in its first approximation without considering the disturbing forces, as a two-body problem: Sun–planet, Earth–satellite, etc. The two-body problem is a well-known integrable problem and its solution is given by a set of the orbital elements $\vec{\sigma}$ [1–3]. The most common set of elements

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is the third set of Brower and Clemence $\vec{\sigma} = (a, e, i, \Omega, \omega, M)$ [3], $M = n(t - t_0) + M_0$, where $n = \mu^{1/2}a^{-3/2}$ is the mean motion and M_0 is the mean anomaly in the initial epoch.

To solve the motion problem there are two main ways: the analytical methods and the numerical methods. Analytical methods to solve the perturbed problem can be appropriate in case of the study of the planetary motion. In this case, the description of the perturbed motion can be obtained by means of the perturbation theory. This method is based on the Lagrange variation of constants and for each planet the variation of the elements is described by the Lagrange planetary equations [4]

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \sigma} \\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial \sigma} \\
\frac{di}{dt} &= -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} + \frac{\operatorname{ctg} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{d\varepsilon}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e},
\end{aligned} \tag{2}$$

where ε is a new variable defined by means of the equation:

$$M = \varepsilon + \int_{t_0}^{t} n\, dt. \tag{3}$$

This new variable coincides with M_0 in the case of the unperturbed motion.

The disturbing potential R due to the disturbing bodies $i = 1, \ldots, N$ is given by [4]

$$R = \sum_{k=1}^{N} Gm_k \left[\left(\frac{1}{\Delta_k} \right) - \frac{x \cdot x_k + y \cdot y_k + z \cdot z_k}{r_k^3} \right] \tag{4}$$

where $\vec{r} = (x, y, z)$ are the coordinates of the secondary with respect to the primary and $\vec{r}_k = (x_k, y_k, z_k)$ the position of the kth disturbing body with respect to the primary, Δ_k is the distance between the secondary and the disturbing body k, and m_k the mass of the k-body.

Let us define the orbital coordinates (ξ, η), where $\xi = \overline{ON}$, O is the main focus of the ellipse related to the position of the primary, N the orthogonal projection of the secondary P on the major axis of the ellipse and $\eta = \overline{NP}$. The sign of the coordinates (ξ, η) is defined by the position in its orbit; both of them are related to the true anomaly f, the secondary anomaly f' which is the angle between the secondary and the periapsis from the secondary focus of the ellipse, and g is the eccentric anomaly [3,2,1]

$$\xi = r\cos f = a(\cos g - e), \qquad \eta = r\sin f = a\sqrt{1-e^2}\sin g, \tag{5}$$

$$r = \frac{a(1-e^2)}{1+e\cos f} = a(1 - e\cos g), \qquad r' = \frac{a(1-e^2)}{1+e\cos f'} = a(1 + e\cos g), \tag{6}$$

where r the radius vector and $r' = 2a - r$ the radius vector of P with respect to the secondary focus of the ellipse. The eccentric anomaly g is connected with the mean anomaly M through Kepler's equation

$$M = g - e\sin g. \tag{7}$$

To solve the problem with analytical methods using the mean anomalies as temporal variables, it is necessary to obtain the analytic development of the main quantities of the two-body problem as Fourier series of the mean anomalies. These developments can be very long if the value of the eccentricity is not small. In order to improve the convergence of the series, Nacozy [5] extends the concept of partial anomalies introduced in 1856 by Hansen; these anomalies are defined in several regions of the orbit and the convergence of the series is improved by choosing an appropriate anomaly for each orbital region.

Nacozy [6] generalizes the transformation $dt = Crdr$ introduced by Sundman in 1912 to regularize the origin of the three-body problem defining a new variable, called intermediate anomaly, as $dt = \mu^{1/2}r^{3/2}d\tau$. Unfortunately, this variable is not normalized in $[0, 2\pi]$ on the orbit. This variable can be normalized as τ^* defined by $dM = \frac{2}{\pi}\frac{K\left(\sqrt{-1}\sqrt{\frac{2e}{1-e}}\right)}{\sqrt{1-e}}\left(\frac{r}{a}\right)^{3/2}d\tau^*$ where $K(x)$ is the complete elliptic integral of first kind.

Janin and Bond [7] define a new one-parametric family of anomalies Ψ_α, called Generalized Sundman anomalies, defined as $C_\alpha(e)r^\alpha d\Psi_\alpha = dM$.