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Stability of King's family of iterative methods with memory[☆]

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ABSTRACT

In the literature exist many iterative methods with memory for solving nonlinear equations, the most of them designed in the last years. As they use the information of (at least) the two previous iterates to generate the new one, usual techniques of complex dynamics are not useful in this case. In this paper, we present some real multidimensional dynamical tools to undertake this task, applied on a very well-known family of iterative schemes; King's class. It is showed that the most of elements of this class present a very stable behavior, visualized in different dynamical planes. However, pathological cases as attracting strange fixed points or periodic orbits can also be found.

1. Introduction

In the last decades, iterative methods for solving nonlinear equations have proved their usefulness in many branches of Science and Technology. Many of them are designed almost ad-hoc, for solving specific types of problems, like derivative-free schemes, for those problems that do not allow to calculate the derivative of the nonlinear equation to be solved, usually because it does not have an explicit expression, or it is too expensive (in the computational sense of the term) to calculate it.

In this work, we use the dynamical tools presented in [1] on iterative schemes with memory for solving nonlinear equations. The design of this kind of methods has experimented an important growth in the last years, the early works of Traub [2], later developed by Petković et al. [3–5] and used by other authors (see [6–9] and references inside), but the understanding of their stability has not been developed. Nevertheless, as the fixed point iteration functions have more than one variable, it is necessary to use some specific dynamical elements joint with some auxiliary functions to facilitate the calculations. Also some dynamical concepts have been adapted to achieve the appropriate numerical sense.

Let us consider the problem of finding a simple zero of a function $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, that is, a solution $\alpha \in D$ of the nonlinear equation $f(x) = 0$. If an iterative method with memory is employed (specifically, one that uses two previous iterations to calculate the following estimation), whose iterative expression is

$$x_{k+1} = g(x_{k-1}, x_k), \quad k \geq 1,$$

where x_0 and x_1 are the initial estimations, a fixed point will be obtained when $x_{k+1} = x_k$, that is, $g(x_{k-1}, x_k) = x_k$. Now, this solution can be obtained as a fixed point of the function $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by means of the fixed-point iteration method

$$\begin{aligned} G(x_{k-1}, x_k) &= (x_k, x_{k+1}), \\ &= (x_k, g(x_{k-1}, x_k)), \quad k = 1, 2, \ldots, \end{aligned}$$

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being x_0 and x_1 the initial estimations. So, we will state that (x_{k-1}, x_k) is a fixed point of G if

$$G\left(x_{k-1}, x_k\right) = \left(x_{k-1}, x_k\right).$$

So, not only $x_{k+1} = x_k$, but also $x_{k-1} = x_k$ by definition of G. Besides, $x^* \in \mathbb{R}^2$ is a k-periodic point if $G^k\left(x^*\right) = x^*$ and $G^p\left(x^*\right) \neq x^*$, for $p = 1, 2, \ldots, k-1$.

We will analyze the local convergence of each one of the methods with memory under study. To get this aim, we will use the following result, that can be found in [10], where e_k denotes the error at the kth iteration, $e_k = x_k - \alpha$.

Theorem 1. *Let ψ be an iterative method with memory that generates a sequence $\{x_k\}$ of approximations to the root α, and let this sequence converge to α. If there exist a nonzero constant η and nonnegative numbers t_i, $i = 0, 1, \ldots, m$, such that the inequality*

$$|e_{k+1}| \leq \eta \prod_{i=0}^{m} |e_{k-i}|^{t_i}$$

holds, then the R-order of convergence of the iterative method ψ satisfies the inequality

$$O_R(\psi, \alpha) \geq s^*,$$

where s^* is the unique positive root of the equation

$$s^{m+1} - \sum_{i=0}^{m} t_i s^{m-i} = 0.$$

In order to analyze the dynamical behavior of a fixed-point iterative method with memory for nonlinear equations on a polynomial $p(z)$, it is necessary to recall some basic dynamical concepts.

Let us denote by $G(z)$ the vector-valued fixed-point function associated to an iterative method with memory on the scalar polynomial $p(z)$. Let us note that the following concepts and results are also valid when the iterative method is applied on a general function $f(z)$.

Definition 1. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector function. The orbit of a point $x^* \in \mathbb{R}^2$ is defined as the set of successive images of x^* by the vector function G, $\{x^*, G(x^*), \ldots, G^m(x^*), \ldots\}$.

The dynamical behavior of the orbit of a point of $\mathbb{R}^2$ can be classified depending on its asymptotic behavior. In this way, we will consider that a point $(z, x) \in \mathbb{R}^2$ is a fixed point of G if $G(z, x) = (z, x)$.

Moreover, as the concept of critical point corresponds to any that makes singular the Jacobian matrix associated to fixed point operator, we will state that a point $x_c \in \mathbb{R}^2$ is a critical point of G if $det(G'(x_c)) = 0$. Indeed, if a critical point is not (r_i, r_i), $i = 1, \ldots, n$ where r_i, $i = 1, \ldots, n$ are the roots of $p(z)$, it will be called free critical point. On the other hand, if a fixed point (z, x) is different from (r_i, r_i), $i = 1 \ldots, n$ where r_i, $i = 1 \ldots, n$ are the roots of $p(z)$, it is called strange fixed point and its character can be analyzed in the same manner.

We recall a known result in Discrete Dynamics that gives the stability of fixed points for multivariable nonlinear operators.

Theorem 2 *([11, page 558]). Let G from $\mathbb{R}^n$ to $\mathbb{R}^n$ be $\mathcal{C}^2$. Assume x^* is a k-periodic point. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $G'(x^*)$.*

(a) *If all the eigenvalues λ_j have $|\lambda_j| < 1$, then x^* is attracting.*
(b) *If one eigenvalue λ_{j_0} has $|\lambda_{j_0}| > 1$, then x^* is unstable, that is, repelling or saddle.*
(c) *If all the eigenvalues λ_j have $|\lambda_j| > 1$, then x^* is repelling.*

In addition, a fixed point is called hyperbolic if all the eigenvalues λ_j of $G'(x^*)$ have $|\lambda_j| \neq 1$. In particular, if there exist an eigenvalue λ_i such that $|\lambda_i| < 1$ and an eigenvalue λ_j such that $|\lambda_j| > 1$, the hyperbolic point is called a saddle point.

Then, if x^* is an attracting fixed point of the rational function G, its basin of attraction $\mathcal{A}(x^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(x^*) = \left\{x_0 \in \mathbb{R}^n : G^m(x_0) \to x^*, m \to \infty\right\}.$$

The rest of the paper is organized as follows: Section 2 is devoted to the construction of a low-order variant with memory of King's family of iterative methods. The real multidimensional discrete dynamics on this class of schemes is made in Section 3 and some dynamical planes, covering the stable and unstable behavior showed in the previous section, are presented in Section 4. Finally, some conclusions and future works are stated.

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