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A unified approach for the development of k-step block Falkner-type methods for solving general second-order initial-value problems in ODEs

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ABSTRACT

A family of k-step block multistep methods where the main formulas are of Falkner type is proposed for the direct integration of the general second order initial-value problem where the differential equation is of the general form $y'' = f(x, y, y')$. The two main Falkner formulas and the additional ones to complete the block procedure are obtained from a continuous approximation derived via interpolation and collocation at $k+1$ points. The main characteristics of the methods are discussed through their formulation in vector form. Each method is formulated as a group of $2k$ simultaneous formulas over k non-overlapping intervals. In this way, the method produces the approximation of the solution simultaneously at k points on these intervals. As in other block methods, there is no need of other procedures to provide starting approximations, and thus the methods are self-starting (sharing this advantage of Runge–Kutta methods). The resulting family is efficient and competitive compared with other existing methods in the literature, as may be seen from the numerical results.

1. Introduction

Second-order differential equations appear frequently in the applied sciences. Examples of that are the mass movement under the action of a force, problems of orbital dynamics, electrical circuits, mechanical oscillators, or in general, any problem involving Newton's second law.

Our goal is to obtain a numerical solution for a second-order initial value problem (I.V.P.) of the general form

$$y''(x) = f(x, y(x), y'(x)), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0. \tag{1}$$

Although it is possible to integrate a second-order I.V.P. by reducing it to a first-order system and applying one of the methods available for such systems, it seems more natural to provide numerical methods in order to integrate the problem directly. The advantage of this procedure lies in the fact that they are able to exploit special information about ODEs, resulting in an increase in efficiency. For instance, it is well-known that Runge–Kutta–Nyström methods for (1) involve

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a real improvement as compared to standard Runge–Kutta methods for a given number of stages [1, p. 285], although the computational cost remains high because of the number of function evaluations. On the other hand, a linear k-step method for first-order ODEs becomes a $2k$-step method for (1), [1, p. 461], increasing the computational work. In the words of F.T. Krogh [2], "the direct integration of second order systems requires about half the number of function evaluations required for integrating the equivalent first order system".

Among the general procedures for direct integration of the problem in (1), the Falkner methods [3,4] are a class of schemes that may be used for this purpose. Other approach recently used for the direct integration of (1) consists in the block formulation. This article combines these two approaches, and presents a block formulation on Falkner methods'. After considering a continuous approximation to the solution by a polynomial approach, the block method is derived via interpolation and collocation procedures. In this way, two main formulas are obtained, which are the Falkner classical method, and other additional formulas to complete the block scheme.

The paper is organized as follows. In Section 2, we present a review of classical Falkner methods. In Section 3, after obtaining a continuous approximation for the exact solution $y(x)$, this is used to generate the two main discrete formulas and the additional methods for solving the problem in (1). The analysis of the methods is discussed in Section 4. Section 5 is devoted to present some particular cases of the general block Falkner method which have been presented previously. Implementation details are considered in Section 6. Some numerical examples are given in Section 7 to show the efficiency of the proposed methods. Finally, some conclusions of the paper are discussed in Section 8.

2. Review of Falkner methods

The explicit Falkner method of k steps consists in a couple of formulas that can be written in the form (see [5])

$$y_{n+1} = y_n + h\, y'_n + h^2 \sum_{j=0}^{k-1} \beta_j \, \nabla^j f_n, \tag{2}$$

$$y'_{n+1} = y'_n + h \sum_{j=0}^{k-1} \gamma_j \, \nabla^j f_n, \tag{3}$$

where h is the stepsize, y_n and y'_n are approximations to the values of the solution and its derivative at $x_n = x_0 + n\,h$, $f_n = f(x_n, y_n)$, and $\nabla^j f_n$ is the standard notation for the backward differences. The coefficients β_j and γ_j can be obtained using the generating functions

$$G_\beta(t) = \sum_{j=0}^{\infty} \beta_j \, t^j = \frac{t + (1-t)\,\mathrm{Log}(1-t)}{(1-t)\mathrm{Log}^2(1-t)},$$

$$G_\gamma(t) = \sum_{j=0}^{\infty} \gamma_j \, t^j = \frac{-t}{(1-t)\mathrm{Log}(1-t)},$$

which have been obtained similarly as that for the Störmer or Cowell methods (see [6, p. 291]).

The implicit Falkner method of k steps consists in two formulas that can be written as (see [5])

$$y_{n+1} = y_n + h\, y'_n + h^2 \sum_{j=0}^{k} \beta^*_j \, \nabla^j f_{n+1}, \tag{4}$$

$$y'_{n+1} = y'_n + h \sum_{j=0}^{k} \gamma^*_j \, \nabla^j f_{n+1}, \tag{5}$$

with generating functions for the coefficients given respectively by

$$G_{\beta^*}(t) = \sum_{j=0}^{\infty} \beta^*_j \, t^j = \frac{t + (1-t)\,\mathrm{Log}(1-t)}{\mathrm{Log}^2(1-t)},$$

$$G_{\gamma^*}(t) = \sum_{j=0}^{\infty} \gamma^*_j \, t^j = \frac{-t}{\mathrm{Log}(1-t)}.$$

Note that the formulas in (3) and (5) are respectively the Adams–Bashforth and Adams–Moulton schemes for the problem $(y')' = f(x, y)$, which are used to follow the values of the derivative. In the Appendix we have included the implicit Falkner formulas for the solution and the derivative from $k = 1$ up to $k = 6$.

All the above formulas are of multistep type, specifically k-step formulas, and so k previous values must be provided in order to proceed with the methods (the Runge–Kutta methods are commonly used to obtain the starting values).

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