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A first approach in solving initial-value problems in ODEs by elliptic fitting methods

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ABSTRACT

Exponentially-fitted and trigonometrically-fitted methods have a long successful history in the solution of initial-value problems, but other functions might be considered in adapted methods. Specifically, this paper aims at the derivation of a new numerical scheme for approximating initial value problems of ordinary differential equations using elliptic functions. The example considered is the undamped Duffing equation where the forcing term is of autonomous type affected by a perturbation parameter. The new scheme is constructed by considering a suitable approximation to the theoretical solution based on elliptic functions. The proposed elliptic fitting procedure has been tested on a variety of problems, showing its good performance.

1. Introduction

Consider the following IVP given by

$$x'' + ax + bx^3 = \epsilon\, g(x), \qquad x(t_0) = x_0, \qquad x'(t_0) = 0, \quad t \in [0, t_N], \tag{1}$$

which is a particular form of the undamped Duffing equation with forcing term $f(x) = \epsilon\, g(x)$. We assume that $a, b > 0$, which may be referred as a hardening stiffness system [1]. An equation of this type arises for example in the non-linear vibration of beams/plates subjected to axial/membrane loading.

It is well-known that an initial value problem of the form in (1) can be solved analytically just in a few cases (see [2–5]). Closed-form solutions to Eq. (1) for a general forcing function are not known. We are interested in finding an approximate discrete solution, say $x_n \simeq x(t_n)$, on the nodal points $t_n = nh;\ n = 0, 1, 2, \ldots, N$, where h is called the *step length*. Nonlinear oscillations are of great importance in physical science, mechanical structures and other engineering problems.

In the present paper, we present a new scheme for the numerical integration of the initial value problem (1). The proposed scheme has second order convergence. Numerical results are presented to validate the efficiency of the proposed method.

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2. Basic preliminaries

The differential problem

$$x'' + ax + bx^3 = 0, \qquad x(t_0) = x_0, \qquad x'(t_0) = 0, \tag{2}$$

where $a, b > 0$ are two parameters, is a particular form of the Duffing oscillator where the restoring force is characterized by a linear term plus a cubic nonlinear term. In this equation x stands for the displacement from the equilibrium position while the forcing strength is null.

In this case we have a single-well anharmonic potential $V(x)$ given by

$$V(x) = \frac{a}{2}x^2 + \frac{b}{4}x^4$$

and the true solution is given by the elliptic function $cn(t, k)$ where t denotes a real variable and k is the modulus, which depends on the initial conditions. Assuming a mass of unity, the constant total energy function is given by

$$H(x, x') = \frac{1}{2}(x')^2 + V(x) = E.$$

From this equation we can derive the period of oscillation T which is given by (see [6])

$$T = \int_{x_1}^{x_2} \sqrt{\frac{2}{E - V(x)}}\, dx$$

where x_1 and x_2 are such that $V(x_1) = V(x_2) = E$.

In view of the true solution of the problem in (2) it makes sense that in order to obtain an approximate solution of the problem in (1), we consider the Jacobi elliptic function cn. The main motivation in using such elliptic function comes from the fact that it is the exact solution of the nonlinear oscillator in (2), and thus we will derive an elliptic fitting method for solving the problem in (1).

3. The new scheme

In order to solve the problem in (1) we propose the following approximation to the theoretical solution $x(t)$ of (1) at $t = t_n + h$

$$x_{n+1} = \frac{1}{6}h^2(4f_n - f_{n-1}) + \frac{2}{D}\left(w^2 x_n \phi(h) - x'_n \phi'(h)\right) \tag{3}$$

$$\begin{aligned} x'_{n+1} &= \frac{1}{2}h(3f_n - f_{n-1}) \\ &= +\frac{2w^2}{D^2}\left(x_n \phi'(h)\left(2a + b\,s + b\,d\phi(h)^2\right) + x'_n \phi(h)\left(2a - b\,d + s\,b\,\phi(h)^2\right)\right) \end{aligned}$$

where we have considered the function $f(x) = \epsilon\, g(x)$, and as it is usual we set $f_n = f(x_n), f_{n-1} = f(x_{n-1})$. The function $\phi(t)$ has been taken as $\phi(t) = cn(wt, m)$ with

$$w = \sqrt{a + b\,x_0^2}, \qquad m = \frac{b\,x_0^2}{2w^2}$$

and the other parameters appearing in the method are given by

$$d = x_n^2 - x_0^2, \qquad s = x_n^2 + x_0^2, \qquad D = 2a + b\,s - d\,b\,\phi(h)^2.$$

Note that the above method consists in two formulas, one to follow the solution and another one to follow the derivative.

After some algebra, it can be shown that the numerical scheme in (3) is exact for solving the problem in (1) when $\epsilon = 0$. We can state this in a more rigorous form by means of the following statement.

Proposition. *The elliptic fitting method in (3) is exact, except round-off errors, for solving the initial-value problem*

$$x'' + ax + bx^3 = 0, \qquad x(t_0) = x_0, \qquad x'(t_0) = 0, \quad t \in [0, t_N].$$

In order to obtain the algebraic order of the above method we consider the following differential operator associated to the equation in (3)

$$\begin{aligned} L[z(t); h] = z(t+h) &- \frac{1}{6}h^2\left(4\left(z''(t) + az(t) + bz(t)^3\right) - \left(z''(t-h) + az(t-h) + bz(t-h)^3\right)\right) \\ &- \frac{2\left(w^2\phi(h)z(t) - \phi'(h)z'(t)\right)}{2a + b\left(z(t)^2 + z_0^2\right) - b\phi(h)^2\left(z(t)^2 - z_0^2\right)}. \end{aligned} \tag{4}$$

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