ARTICLE IN PRESS
Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A convergent and dynamically consistent finite-difference method to approximate the positive and bounded solutions of the classical Burgers–Fisher equation

J.E. Macías-Díaz, A.E. González *

Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Avenida Universidad 940, Ciudad Universitaria, Aguascalientes 20131, Mexico

ARTICLE INFO

Article history:
Received 21 July 2015

MSC:
65Q10
65M06
65Q30
35K55

Keywords:
Burgers–Fisher equation
Existence and uniqueness
Convergence analysis
Exact method
Positivity and boundedness
Nonlinear finite-difference scheme

ABSTRACT

In this work, we investigate both analytically and numerically a spatially two-dimensional advection–diffusion–reaction equation that generalizes the Burgers' and the Fisher's equations. The partial differential equation of interest is a nonlinear model for which the existence and the uniqueness of positive and bounded solutions are analytically established here. At the same time, we propose an exact finite-difference discretization of the Burgers–Fisher model of interest and show that, as the continuous counterpart, the method proposed is capable of preserving the positivity and the boundedness of the numerical approximations as well as the temporal and spatial monotonicity of the discrete initial–boundary conditions. It is shown that the method is convergent with first order in time and second order in space. We provide some simulations that illustrate the fact that the proposed technique preserves the positivity, the boundedness and the monotonicity.

1. Introduction

The fields of numerical analysis and the theory of approximation have witnessed the rise of many approaches in the development of techniques to approximate the solution of partial differential equations. Most of these methodologies have been proposed in order to estimate the solutions of the continuous model of interest with certain degree of accuracy in mind. Stability has been also an important characteristic in the design of new numerical techniques. However, most of the methodologies developed in these areas have frequently forgotten to take into account specific mathematical features of the solutions of the model of interest. Nevertheless, in recent years Ronald E. Mickens expressed the need to develop approximation techniques that take into consideration the physical and mathematical properties of the solutions [1], and not only the numerical characteristics. Mickens called this feature *dynamical consistency*, and urged for its consideration in the design of novel methods of approximation. Obviously, he was not the first researcher to work on the development of dynamically consistent numerical methods to approximate the solutions of mathematical models; however, he noticed that this perspective was itself an important avenue of research. Indeed, before his seminal work, many numerical techniques that preserve the energy of systems had been reported [2–4].

It is important to recall that the development of conservative finite-difference schemes for physical models was a fruitful topic of investigation. Naturally, if a physical system proves to conserve the total energy throughout time, one

* Corresponding author. Tel.: +52 449 910 8400; fax: +52 449 910 8401.
E-mail addresses: jemacias@correo.uaa.mx (J.E. Macías-Díaz), aegonzal@correo.uaa.mx (A.E. González).

http://dx.doi.org/10.1016/j.cam.2015.11.018

expects for a numerical integrator to preserve the discrete energy, also. In similar fashion, any computational technique to estimate the solutions of a momentum- and mass-preserving system should also preserve the discrete momentum and total mass [5]. More recently, several other characteristics of solutions have been taken into consideration. For example, many computational methods guarantee that, under suitable circumstances, initial–boundary conditions which are positive and bounded will yield positive and bounded successive approximations [6–10]. It is worthwhile to mention that the preservation of each of these two properties is physically relevant in those phenomena where the variables of interest are measured in absolute scales, and where natural limitations exist in the environment. One may think of the dynamics of populations [11,12], in which the variable of interest is population size or population density. Another example is the dynamics of the temperature of a system measured in Kelvin [13,14].

In the present manuscript, we consider the two-dimensional classical Burgers–Fisher equation as the model of interest. This equation is a generalization of the classical Fisher's equation which was investigated in 1937 simultaneously and independently by R. A. Fisher [15] and A. Kolmogorov, I. Petrovsky and N. Piscounov [16]. Our model also considers the presence of an advection/convection term of the Burgers type [17]. As we will see in this work, the Burgers–Fisher equation under investigation possesses classical solutions which are positive and strictly bounded from above by 1. We will propose here a nonlinear finite-difference discretization that is

1. nonlinear,
2. convergent,
3. monotone,
4. stable,
5. exact,
6. positivity-preserving,
7. boundedness-preserving,
8. asymptotically consistent and
9. computationally fast.

More precisely, the technique presented in this work will be a fast nonlinear technique for which the positive solutions are given in an exact form. Moreover, the method is convergent of first order in time and second order in space. We will establish the fact that our scheme preserves monotonicity whence the property of stability will be derived. In addition, our technique preserves the positivity and the boundedness of approximations, in agreement with the properties of the continuous model. Furthermore, we will see that the constant solutions of the continuous model are also constant solutions of the finite-difference scheme (this is what we mean by *asymptotic consistency*).

This work is divided as follows. In Section 2, we describe the continuous model of interest, namely, the classical Burgers–Fisher equation in two spatial dimensions. We derive therein some results that guarantee the existence and uniqueness of positive and bounded classical solutions. For computational comparisons, we recall the expression of a traveling-wave solution of the continuous model. Section 3 is devoted to introducing the finite-difference scheme to approximate the solutions of the model of interest. We provide therein an alternative formulation of the method, which leads to establishing the fact that the positive solutions of the finite-difference scheme may be calculated exactly. The most important properties of our methodology are established in Section 4. More precisely, we establish conditions under which the finite-difference scheme is convergent, and that it preserves the positivity, the boundedness and the monotonicity of initial approximations. Section 5 is devoted to providing qualitative comparisons of the numerical approximations obtained through our method against the traveling-wave solution of the Burgers–Fisher equation. Quantitative numerical results summarized in that section confirm the convergence properties of the finite-difference scheme. This manuscript closes with a section of concluding remarks.

2. Continuous model

2.1. Burgers–Fisher equation

Throughout this work, we let α and T be real numbers with $T > 0$. We will use the nomenclature $\overline{\mathbb{R}^+}$ to represent the set of nonnegative real numbers. Throughout, we will consider an open and connected domain $\Omega \subseteq \mathbb{R}^2$, and we will suppose that u is a real function defined on $\overline{\Omega} \times [0, T]$ which satisfies the nonlinear $(2+1)$-dimensional partial differential equation

$$\frac{\partial u}{\partial t} + \alpha u \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) - \Delta u - f(u) = 0, \tag{1}$$

for every $(\mathbf{x}, t) \in \Omega \times (0, T)$. The parameter α represents the advection/convection coefficient, and we consider a reaction term of the form

$$f(u) = u(1-u). \tag{2}$$

This model is a generalized form of both the Burgers' and the Fisher's equations, and we will refer to it simply as the *Burgers–Fisher equation*. The first step in our investigation will be to establish the existence and uniqueness of positive and bounded solutions.

Please cite this article in press as: J.E. Macías-Díaz, A.E. González, A convergent and dynamically consistent finite-difference method to approximate the positive and bounded solutions of the classical Burgers–Fisher equation, Journal of Computational and Applied Mathematics (2015), http://dx.doi.org/10.1016/j.cam.2015.11.018