Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

New answers to an old question in the theory of differential–algebraic equations: Essential underlying ODE versus inherent ODE

Roswitha März

Humboldt University of Berlin, Institute of Mathematics, D-10099 Berlin, Germany

ARTICLE INFO

Article history:
Received 15 December 2015
Received in revised form 29 August 2016

Keywords:
Differential–algebraic equation
Essential underlying ODE
Inherent ODE
Adjoint pairs
Lagrange identity
Lyapunov spectrum
Lyapunov regularity

ABSTRACT

In the context of linear differential–algebraic equations (DAEs) one finds different associated explicit ordinary differential equations (ODEs), among them *essential underlying* and *inherent explicit regular* ones, abbreviated: EUODEs and IERODEs. EUODEs have been introduced in 1991 for index-2 DAEs in Hessenberg form by means of special transformations. IERODEs result within the framework of the projector based decoupling. Each such explicit ODE is occasionally considered to rule the flow of the DAE.

The question to which extend EUODEs and IERODEs are related to each other has been asked promptly after 1991. For index-2 Hessenberg-form DAEs, answers have been given in 2005, saying that EUODEs represent somehow condensed IERODEs. Recently, EUODEs have been indicated for general arbitrary-index DAEs and it has been proved that they are condensed IERODEs. The understanding of the relation between the IERODE and the EUODEs enables to uncover the stability behavior of the DAE flow.

In the present paper we show that both, the IERODEs and EUODEs of a DAE with arbitrary high index do not at all depend on derivatives of the right-hand side. We consider adjoint pairs of DAEs and provide generalizations of the classical Lagrange identity. Furthermore, we address Lyapunov spectra and Lyapunov regularity.

1. Introduction

In the context of linear differential–algebraic equations (DAEs) one finds different associated explicit ordinary differential equations (ODEs), among them *essential underlying* ODEs (EUODEs) and *inherent explicit regular* ODEs (IERODEs). EUODEs have been distinguished by Ascher and Petzold [1] for index-2 DAEs in Hessenberg form by means of special transformations. IERODEs result within the framework of the projector based decoupling.

Both EUODEs and IERODEs are occasionally considered to rule the flow of the DAE. How are they related to each other? This question has been asked promptly after 1991. First answers have been given by Balla and Vu Hoang Linh [2,3] pointing out that, for index-2 Hessenberg-form DAEs and general index-1 DAEs, an EUODE represents a condensed IERODE. Recently, EUODEs associated with general arbitrary-index DAEs have been distinguished in [4]. Also in the general case, the EUODEs can be seen as condensed IERODEs.

Any regular linear differential–algebraic equation with properly stated leading term features a unique IERODE living in the given space. In contrast, there are several EUODES living in a transformed space with possible minimal dimension. We show that the IERODEs and EUODEs are the only associated explicit ODEs which do not at all depend on derivatives of

E-mail address: maerz@math.hu-berlin.de.

http://dx.doi.org/10.1016/j.cam.2016.09.039

Please cite this article in press as: R. März, New answers to an old question in the theory of differential–algebraic equations: Essential underlying ODE versus inherent ODE, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.09.039

the right-hand side. This offers the background of a preciser sensitivity analysis and allows to state boundary conditions accurately.

The understanding of the relation between the IERODE and the EUODEs enables to uncover stability properties of the DAE flow such as Lyapunov regularity etc. We particularly concentrate on relevant versions of the Lagrange identity. This should be helpful in the context of numerical boundary value problems, sensitivity analysis, optimization, and control problems, see [1,5–10].

We investigate DAEs with properly involved derivative

$$A(t)(Dx)'(t) + B(t)x(t) = q(t), \quad t \in \mathcal{I} \tag{1}$$

and standard form DAEs

$$E(t)x'(t) + F(t)x(t) = q(t), \quad t \in \mathcal{I}, \tag{2}$$

with coefficients

$$E, F \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),$$
$$A \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)), \qquad D \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^n)), \qquad B \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),$$

with $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ in the real and complex versions, respectively. The interval $\mathcal{I} \subseteq \mathbb{R}$ is arbitrary, possibly infinite. We drop the argument t whenever reasonable. Then the given relations are meant pointwise for all $t \in \mathcal{I}$.

The paper is arranged as follows. In Section 2 we recap known results for index-2 DAEs in Hessenberg form already by means of the notation of the general approach in the following part in order to motivate this approach and to explicate the questions to be considered. In Section 3 we describe the structure of a regular DAE (1). In particular, we deal with its IERODE and EUODEs and show that these are the only associated q-derivative-free explicit ODEs. Adjoint pairs of DAEs are considered in Section 4. We show that, even though the IERODEs of the original DAE and its adjoint equation are not necessarily adjoint to each other, the solutions associated to the DAEs satisfy a Lagrange identity. We specify the results for standard form DAEs in Section 5. General conclusions are drawn in Section 6.

2. Recapping index-2 Hessenberg systems

The system comprising the $m = m_1 + m_2$ equations

$$x_1' + B_{11}x_1 + B_{12}x_2 = q_1, \tag{3}$$
$$B_{21}x_1 = q_2, \tag{4}$$

is said to be a DAE in Hessenberg form of size 2, if the product $B_{21}B_{12}$ remains everywhere nonsingular. This DAE is known to have differentiation index 2 as well as tractability index 2. Writing

$$\underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{=A} \left(\underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{=D} x \right)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix} x = q, \tag{5}$$

one puts the DAE in the form (1). Owing to the nonsingularity of the product $B_{21}B_{12}$, the direct sum decomposition

$$\ker B_{21} \oplus \operatorname{im} B_{12} = \mathbb{K}^{m_1} \tag{6}$$

is valid and the projector-valued function Ω given by

$$\Omega := B_{12}B_{12}^-, \quad B_{12}^- := (B_{21}B_{12})^{-1}B_{21}, \tag{7}$$

projects pointwise onto $\operatorname{im} B_{12}$ along $\ker B_{21}$.

The usual approach to DAEs consists in extracting explicit ODEs with respect to x or x_1 from a derivative array system. The resulting so-called *underlying ODEs* depend on the special way they are provided. For instance, differentiating Eq. (4) and then replacing the derivative x_1' by (3) results in the equation

$$\mathfrak{A}x_1 - \mathfrak{B}x_2 = q_2' - B_{21}q_1, \tag{8}$$

with $\mathfrak{A} := -B_{21}B_{11} + B_{21}'$, $\mathfrak{B} := B_{21}B_{12}$. Differentiating now (8), replacing again the derivative x_1' by (3), multiplying by $\mathfrak{B}^{-1}$ and rearranging terms finally leads to the ODE

$$x_1' = -B_{11}x_1 - B_{12}x_2 + q_1,$$
$$x_2' = -\mathfrak{B}^{-1}(\mathfrak{A}B_{11} - \mathfrak{A}')x_1 - \mathfrak{B}^{-1}(\mathfrak{A}B_{12} + \mathfrak{B}')x_2 + \mathfrak{B}^{-1}(\mathfrak{A}\,q_1 - (q_2' - B_{21}q_1)').$$

The DAE flow is embedded into the m-dimensional flow of the underlying ODE. Observe that derivatives of components of q encroach on this ODE.

Please cite this article in press as: R. März, New answers to an old question in the theory of differential–algebraic equations: Essential underlying ODE versus inherent ODE, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.09.039