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Space–time adaptive linearly implicit peer methods for parabolic problems

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ABSTRACT

In this paper a linearly implicit peer method is combined with a multilevel finite element method for the discretization of parabolic partial differential equations. Following the Rothe method it is first discretized in time and then in space. A spatial error estimator based on the hierarchical basis approach is derived. It is shown to be a reliable and efficient estimator up to some small perturbations. The efficiency index of the estimator is shown to be close to the ideal value one for two one-dimensional test problems. Finally we compare the performance of the overall method, based on second, third, and fourth order peer methods with that of some Rosenbrock methods. We conclude that the presented peer methods offer an attractive alternative to Rosenbrock methods in this context.

1. Introduction/motivation

In [1] linearly implicit peer methods were introduced for the numerical solution of time-dependent PDEs. The authors followed the Rothe approach, discretizing first in time with a linearly implicit peer method and then solving the arising linear elliptic problems with the finite element method. This approach was already shown to be efficient for Rosenbrock methods [2]. The combination of peer methods with finite elements was implemented and tested within the software package *KARDOS* [3].

The main advantage of peer methods is that they do not show order reduction when applied to stiff ODEs or PDEs [4], as it is the case for Rosenbrock methods [5]. Furthermore, peer methods have good stability properties in comparison with other multistep methods. There are $A(\alpha)$-stable methods available with an α almost equal 90°.

However, the presented method in [1] is only adaptive in time and fixed spatial grids are used for the whole integration. Many problems like the propagation of a flame front are solved more efficiently using also adaptive spatial grids. In this paper we want to fill this gap.

We combine the linearly implicit peer method within the Rothe approach with a multilevel finite element method. Thus we do not use a fixed spatial grid but we build a sequence of nested finite element spaces at every time step. The nested spaces are constructed adaptively with respect to a local spatial error estimation based on a hierarchical basis.

The paper is organized as follows. In Section 2 we specify the problem setting. Then we give a short overview to linearly implicit peer methods. In Section 4 we present the finite element solution of the arising linear elliptic equations and derive

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the spatial error estimator. The error estimator is then proven to be efficient and robust up to some small perturbations. The time step control is explained in Section 5. We then present two numerical experiments in one spatial dimension concerning the efficiency of the spatial error estimation in Section 6. The performance of the presented method is then compared to Rosenbrock methods for some test problems in two spatial dimensions in Section 7. Finally our results are summarized.

2. Parabolic partial differential equations

We consider the nonlinear initial boundary value problem

$$\begin{aligned} \partial_t y(x,t) &= f(x,t,y(x,t)) && \text{in } \Omega \times (0,T], \\ B(x,t,y(x,t))y(x,t) &= g(x,t,y(x,t)) && \text{on } \partial\Omega \times (0,T], \\ y(x,0) &= y_0(x) && \text{on } \bar{\Omega}. \end{aligned} \tag{1}$$

in the same setting as in [6,2]. $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3, denotes a bounded domain with sufficiently smooth boundary $\partial\Omega$. f is a partial differential operator. The boundary operator B stands for a system of boundary conditions interpreted in the sense of traces. g is a given function and y_0 is the initial condition.

We consider a Gelfand triple of separable Hilbert spaces $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{V}'$ with $\mathcal{V} \overset{ds}{\hookrightarrow} \mathcal{H} \overset{ds}{\hookrightarrow} \mathcal{V}'$. We denote the norm on $\mathcal{H}$ induced by the scalar product $(\cdot,\cdot)$ with $|\cdot|$, the norm on $\mathcal{V}$ induced by the scalar product $((\cdot,\cdot))$ with $\|\cdot\|$, and the dual norm on $\mathcal{V}'$ by $\|\cdot\|_*$. The anti duality between $\mathcal{V}$ and $\mathcal{V}'$ is denoted by $\langle\cdot,\cdot\rangle$.

With the operator $F : (0,T] \times \mathcal{V} \to \mathcal{V}'$ we rewrite (1) as an abstract Cauchy problem of the form

$$\partial_t y(t) = F(t, y(t)), \quad 0 < t \le T, \; y(0) = y_0. \tag{2}$$

We assume that (2) has a unique, temporally smooth solution $y(t)$.

We suppose that F is sufficiently differentiable. We set

$$A(t,w) := -F_y(t,w). \tag{3}$$

We assume that $A(t,w) : \mathcal{V} \to \mathcal{V}'$ is a sectorial operator for $t \in (0,T]$ and $w \in \mathcal{W} \subset \mathcal{V}$. The operator $A(t,w)$ is associated with a sesquilinear form

$$a(t,w;v_1,v_2) = \langle A(t,w)v_1, v_2\rangle, \quad v_1, v_2 \in \mathcal{V}. \tag{4}$$

We assume that for all $w \in \mathcal{W}$ and $t \in [0,T]$ the sesquilinear form $a(t,w;v_1,v_2)$ is continuous

$$|a(t,w;v_1,v_2)| \le M_a \|v_1\|\|v_2\| \quad \forall v_1, \; v_2 \in \mathcal{V} \tag{5}$$

and $\mathcal{V}$-elliptic

$$a(t,w;v_1,v_1) \ge \mu_a \|v_1\|^2, \quad \forall v_1 \in \mathcal{V}, \tag{6}$$

with constants M_a and μ_a independent of $t \ge 0$, w, v_1 and v_2. Furthermore, we require Lipschitz continuity of $t \mapsto A(t, w(t))$ in the $\mathcal{L}(\mathcal{V},\mathcal{V}')$-norm, i.e.

$$\|A(t_2, w(t_2)) - A(t_1, w(t_1))\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le L|t_2 - t_1|, \quad \forall t_1, \; t_2 \in [0,T]. \tag{7}$$

Also we assume for the regularity of the second derivatives of F:

$$\|F_{ty}(t,v)v_1\|_* \le C_1\|v_1\| \qquad \forall v_1 \in \mathcal{V}, \tag{8}$$

$$\|F_{yy}(t,v)[v_1,v_2]\|_* \le C_2\|v_1\|\|v_2\| \qquad \forall v_1, v_2 \in \mathcal{V}, \tag{9}$$

with C_1, C_2 independent of v varying in bounded subsets of $\mathcal{V}$ and $t \in [0,T]$.

By setting $Q(t,v) = F(t,v) + A(t,v)v$ for all $v \in \mathcal{V}$, we can rewrite (2) in the form of a quasilinear Cauchy problem

$$\partial_t y + A(t,y)y = Q(t,y), \quad 0 < t \le T, \; y(0) = y_0. \tag{10}$$

From continuity (5) we deduce that $A(t,v)$ is uniformly bounded and has a uniformly bounded inverse A^{-1} [2]. Furthermore, $\mathcal{V}$-ellipticity (6) implies the existence of constants $M > 0$ and angle $\phi < \frac{\pi}{2}$ such that the resolvent bound

$$\|(\lambda I + A(t,w))^{-1}\|_{\mathcal{L}(\mathcal{V})} \le \frac{M}{1+|\lambda|}$$

holds for all $w \in \mathcal{W}$ and for all $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| \le \pi - \phi$ [2]. This setting is the usual one for differential equations of parabolic type and includes the case of semilinear and quasilinear parabolic equations in two and three space dimensions [6].

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